Math 491, Problem Set #9 (Solutions)

1. (a) *How many different polygonal paths of length n are there that start at the point $(0,0)$ and then take n steps of length 1, such that each step is either rightward, leftward, or upward, and such that no point gets visited more than once? Give an explicit formula.*

   This is the same as the number of strings of length n consisting of the symbols R, L, and U (short for Right, Left, and Up, respectively) such that no R is followed by an L and no L is followed by an R. The associated 1-step transfer matrix

$$\begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}$$

   has characteristic polynomial $(t-1)(t^2-2t-1)$, so the answer is of the form $A+Br^n+Cs^n$ where $r = 1+\sqrt{2}$, $s = 1-\sqrt{2}$, and A, B, C are undetermined coefficients. Using the fact that the number of polygonal paths of the desired kind equals 1, 3, and 7 when n is 0, 1, and 2, respectively, we get $A = 0$, $B = 1+\sqrt{2}$, and $C = 1-\sqrt{2}$, so that the final answer is $\frac{1}{2}((1+\sqrt{2})^{n+1} + (1-\sqrt{2})^{n+1})$.

   To do this in Maple, one might proceed as follows:

```
with(linalg):
m := matrix(3,3,[1,0,1,0,1,1,1,1,1]);
p := charpoly(m,t);
sols := solve(p=0,t);
r := sols[2];
s := sols[3];
ans:=solve({A+B*r+C*s=3,A+B*r^2+C*s^2=7,
        A+B*r^3+C*s^3=17},{A,B,C});
```

   The result is a set whose three elements are equations giving the values of A, C, and B respectively. (Note that the Maple command I used doesn't return the values of the variables in the same order as I specified them! Does anyone know of a variant of my command that doesn't suffer from this defect?) By the way, the command `with(linalg)` only needs to be done once per session.

(b) *If one chooses at random one of the paths of length n described in part (a) (so that each of the length-n paths has an equal chance of being chosen), what is the expected value of the y-coordinate of the last point on the path? Find a constant c so that this expected value is asymptotic to cn.*

An appropriate generating function is $\sum_{n\geq 1}(uM^n v)x^n$, where $u = (1,1,y)$, $v = (1,1,1)^T$ (the transpose), and M is a modified version of the preceding transition matrix in which the 1's that correspond to Up-steps are replaced by y's, so that when we multiply the matrix by itself, obtaining a matrix of polynomials in y, a term equal to y^k corresponds to a path that takes k Up-steps. (After we've expressed this generating function in closed form, we'll be able to differentiate it to get at the information we seek.) The entry y in the vector u occurs because it corresponds to taking a step in the Up direction. The generating function can be written as the sum of the nine entries of $u(\sum(Mx)^n)v = u(I - Mx)^{-1}v$, where the matrix Mx is

$$\begin{pmatrix} x & 0 & xy \\ 0 & x & xy \\ x & x & xy \end{pmatrix}.$$

We can use Maple for this:

```
with(linalg):
Mx := [[x,0,x*y],[0,x,x*y],[x,x,x*y]];
Id := [[1,0,0],[0,1,0],[0,0,1]];
u := [[x,x,x*y]];
v := [[1],[1],[1]];
inv := inverse(Id-Mx);
ans := simplify(multiply(u,inv,v)[1,1]);
```

(Note that for Maple, xy must be written as `x*y`; also note that `Mx` is just an indivisible symbol. Observe that the symbol `I` is reserved for the square root of minus 1. Finally, note that the output of `multiply(u,inv,v)` is a 1-by-1 matrix, not a number; hence the need to extract its 1,1 element with the matrix-entry-extraction operator `[1,1]`.) The answer `ans` turns out to be the

simple expression
$$\frac{2+y+xy}{1-x-xy-x^2y}.$$
If we differentiate this with respect to y and then set $y=1$, we will obtain the generating function in which the coefficient of x^n is the sum of the heights of all the polygonal paths.

```
heights := simplify(subs(y=1,diff(ans,y)));
```

gives
$$\frac{x(1+x)^2}{(1-2x-x^2)^2}.$$
To find the asymptotic behavior of the coefficients of this generating function, use partial fractions over the field generated over the rationals by the square root of 2:

```
convert(heights, parfrac, x, sqrt(2));
```

Unfortunately, Maple gives us an answer in which the denominators of the four terms are of the form $x+a$ instead of $1+bx$, but this is only a minor annoyance. The term that controls the growth rate is the term whose denominator is quadratic and vanishes closest to $x=0$. This is the term
$$\frac{1}{4}\frac{-1+\sqrt{2}}{(x+1-\sqrt{2})^2} = \frac{1+\sqrt{2}}{4}(1-x(1+\sqrt{2}))^{-2}.$$
Now we may apply the binomial theorem with exponent -2: the coefficient of x^n in the preceding generating function equals
$$\frac{1+\sqrt{2}}{4}\binom{-2}{n}(-(1+\sqrt{2}))^n = \frac{1+\sqrt{2}}{4}\binom{n+1}{n}(1+\sqrt{2})^n,$$
which grows like $\frac{n}{4}(1+\sqrt{2})^{n+1}$. The answer to part (a) grows like $\frac{1}{2}(1+\sqrt{2})^{n+1}$, so, taking the ratio, we find that the expected height tends to the limit $n/2$. (There must be a nice way to see this!)