Math 491, Problem Set #17
(due 11/25/03)

1. Let $p(n)$ be the number of unconstrained partitions of n if $n \geq 0$, and 0 othewise, so that

$$p(n) = p(n-1)+p(n-2)-p(n-5)-p(n-7)+p(n-12)+p(n-15)--++\dots$$

for all $n > 0$. Use the recurrence for $p(n)$ to compute the last digit of $p(n)$ for every n between 1 and 1000. Make a conjecture about the relationship between the last digit of n and the last digit of $p(n)$; specifically, make a conjecture about which pairs (n mod 10, $p(n)$ mod 10) occur and which don't.

2. Let $f(0) = 1$ and recursively define $f(n) = f(n-1) + f(n-3) - f(n-6) - f(n-10) + f(n-15) + f(n-21) - - + +\dots$ for all $n > 0$, where terms of the form $f(n-k)$ are to be ignored once $k > n$.

   (a) Since the formal power series $F(q) = \sum_{n \geq 0} f(n)q^n = 1 + q + q^2 + 2q^3 + 3q^4 + 4q^5 + 5q^6 + 7q^7 + \dots$ has constant term 1, we saw in class that it admits a (unique) convergent infinite formal product expansion of the form

   $$(1-q)^{a_1}(1-q^2)^{a_2}(1-q^3)^{a_3}(1-q^4)^{a_4}\cdots$$

   Find a_1 through a_{24}, and conjecture a general rule.

   (b) Assuming that your answer from (a) is correct, prove that for a particular set S of positive integers (which you must find!), $f(n)$ equals the number of partitions of n into parts belonging to S.

   (c) Prove that your conjectures from (a) and (b) are correct, e.g. by using the Jacobi triple product identity

   $$\prod_{n=1}^{\infty}(1-x^{2n})(1+x^{2n-1}z^2)(1+x^{2n-1}z^{-2}) = \sum_{m=-\infty}^{\infty} x^{m^2}z^{2m}$$

   (which you do not need to prove). An equivalent form of the Jacobi triple product identity is

   $$\prod_{i=1}^{\infty}(1+xq^i)(1+x^{-1}q^{i-1})(1-q^i) = \sum_{n=-\infty}^{\infty} q^{n(n+1)/2}x^n.$$