Rotor-router types

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A **rotor-router** network is a directed graph G with a designated vertex called the **source** (from which every vertex of G can be reached) and for each vertex v of positive out-degree some infinite periodic sequence of directed edges $e_1^{(v)}, e_2^{(v)}, \ldots$ emanating from v, which we call the **rotor pattern** at v. Vertices of out-degree 0 are called **targets**; we that from every vertex there is a path to a target. Every rotor-router network determines an infinite sequence of targets, called the **hitting sequence**. We imagine a **chip** traveling through the network, repeatedly starting from the source and traveling until it hits a target, such that the chip travels along edge $e_n^{(v)}$ after its nth visit to v; this constraint uniquely specifies the itinerary of the chip. The hitting sequence consists of the successive targets the chip visits.

It can be shown that the chip will visit the target set infinitely often, so that the hitting sequence is infinite, and that the hitting sequence is periodic. See the article "GLPZ", aka *Local-to-Global Principles for Rotor Walk on Graphs* by Giuliano Giacaglia, Lionel Levine, James Propp and Linda Zayas-Palmer (or rather, send me an email and I'll put you on the preprint list, since we don't have a version of the article ready for distribution yet); or see the slides from my April 2010 MIT Combinatorics Seminar talk *Local-to-Global Phenomena for Rotor-Routing* (`http://jamespropp.org/mitcomb10a.pdf`).

Every rotor pattern, being a periodic sequence (say of period n), determines a partition π of $[n]$ where i and j are in the same block iff $e_i^{(v)} = e_j^{(v)}$; we call π the **type** of the rotor pattern. Such a rotor-type can be represented by the list $f(1), f(2), \ldots, f(n)$ where f is any function from $[n]$ to $\mathbf{N}$ whose sets-of-constancy are precisely the blocks of π. E.g., if the rotor pattern at v is the period-3 sequence $e, e', e', e, e', e', e, e', e', \ldots$ (where e and e' are directed edges emanating from v), we can write its type as $1, 2, 2$ or $2, 1, 1$ or $3, 5, 5$ etc. Note also that $1, 2$ and $1, 2, 1, 2$ denote the same rotor-type, associated with a rotor that alternates between two edges.

Likewise, every hitting sequence, being a periodic sequence (say of period N), determines a partition π of $[N]$, which we call the **type** of the hitting sequence.

Claim 1: Every type of hitting sequence can be achieved by a rotor-router network in which all rotors are of type 1,2. We therefore say that the rotor-type 1,2 is **universal**.

Proof of Claim 1: Consider a hitting sequence whose type is a partition π of $[N]$. Take any k with $2^k \geq N$. We create a rotor-router network whose underlying directed graph is a complete binary tree with 2^k leaves and with rotors of type $1, 2$ at all non-leaf vertices. We modify the network by introducing directed edges from $2^k - N$ of the leaves back to the source (the root) and by making identifications of the remaining N leaves in accordance with the partition π.

Claim 2: The rotor-type 1,2,2 is universal.

Proof of Claim 2: We create a rotor-router network whose underlying directed graph is a complete binary tree with 4 leaves and with rotors of type $1, 2, 2$ at the three non-leaf vertices. We modify the network by introducing directed edges from the leftmost and rightmost leaves back to the source (the root). If the two remaining leaves are denoted by a and b, then the resulting hitting sequence is $a, b, a, b, \ldots$, which is of type 1,2. Since $1, 2$ is universal, we conclude that $1, 2, 2$ is as well.

Claim 3: The rotor-types 1,2,1, and 1,2,2,1 are **not** universal.

Proof of Claim 3: GLPZ shows that if all rotors in a network are palindromic (i.e., each rotor pattern, as a partition of $[n]$ for some n, is invariant under the map $[n] \to [n]$, $k \mapsto n + 1 - k$), then so is the hitting sequence.

Claim 4: The rotor-types 1,1,2,2 and 1,1,1,2,2,2 are **not** universal.

Proof of Claim 4: GLPZ shows that for any $m \geq 1$, if all rotors in a network are m-repetitive (i.e., each rotor pattern, as a partition of $[n]$ for some n, has i and j in the same block whenever $\lceil i/m \rceil = \lceil j/m \rceil$, or equivalently, each rotor pattern, written as a sequence of length n, is a concatenation of constant sequences of length m), then so is the hitting sequence.

Question 1: Which rotor-types are universal? E.g., is 1,1,2,2,2 universal? What about the *pair* of types 1,1,2,2 and 1,1,1,2,2,2, or the pair of types 1,2,1 and 1,1,2,2, or the pair of types 1,2,2,1 and 1,1,2,2?

More generally, call a collection $\mathcal{C}$ of rotor-types a **closed class** if every rotor-router network whose rotors all have types belonging to $\mathcal{C}$ has a hitting sequence whose type belongs to $\mathcal{C}$. E.g., the collection of palindromic rotor-types is closed, as is the collection of k-repetitive rotor-types ($k \geq 1$).

It can be shown that the rotor-type 1,2,1 is universal for the class of palindromic rotor-types, as is the rotor-type 1,2,2,1, and that the rotor-type 1,1,…,1,2,2,…,2 consisting of m 1's and m 2's is universal for the class of m-repetitive rotor-types.

Question 2: What are the closed classes of rotor-types? Is every such class finitely generated? Singly generated?