James Propp

Order and Randomness in Tilings

Abstract: Combinatorics is the branch of mathematics that asks questions that start "In how many ways can we ...?" It began two millennia ago when poet-mathematicians in India counted verse patterns, and it continues to thrive today. My project extends my earlier work on dimers (pairs of linked objects) to the study of polymers (longer chains of linked objects) on grids. These mathematical testbeds are not literal physical models, but they reveal structural phenomena with analogues in statistical physics. By combining drawings, computer simulations, and rigorous proofs, I aim to uncover both the combinatorial structure and probabilistic behavior of such tilings. Equally important is my goal of widening access to mathematics; combinatorics offers a low barrier to entry, making it an ideal gateway for students from diverse backgrounds to experience research and to begin to see themselves as mathematicians.

During my year at Radcliffe, I'll investigate tilings like the one in Figure 1. The figure depicts a region made of sixty small hexagons, grouped into twenty tiles each of which is a "stone" or a "bone" (see Figure 2). Combinatorialists call these systems polymer models.

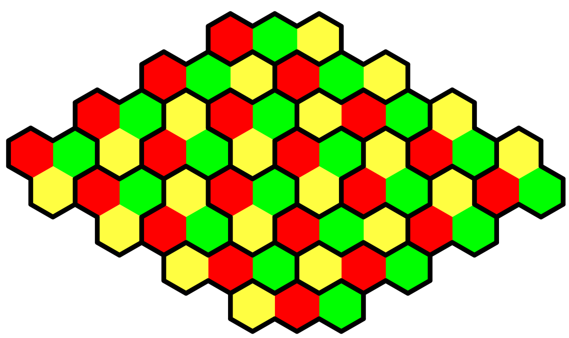

Figure 1: A tiling of a small region by stones and bones.

The region being tiled in Figure 1 is the fourth member of an infinite sequence of similarly-shaped regions R_n ($n = 1, 2, 3, \ldots$). A central combinatorial question is: in how many ways can R_n be tiled with stones and bones? The remarkably simple answer is $2^{n(n+1)/2}$. From the

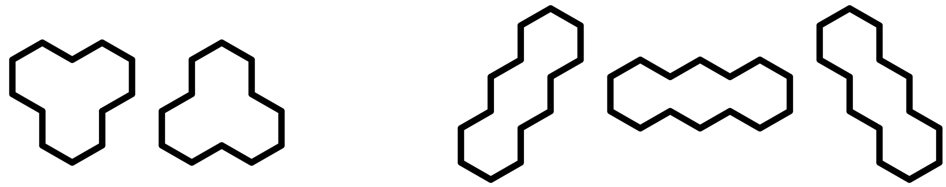

Figure 2: Stones (left) and bones (right).

probabilistic viewpoint, however, the key question is: what does a statistically random tiling look like? Here again a strikingly simple picture emerges: for large n, one will see a circular boundary separating a disordered interior from four orderly corners. Figure 3 illustrates the same “order from randomness” phenomenon for tilings that use dominoes (Figure 4), though the individual tiles in Figure 3 are hard to see. In both cases, order emerges from randomness, with different forms of organization coexisting in different parts of the system.

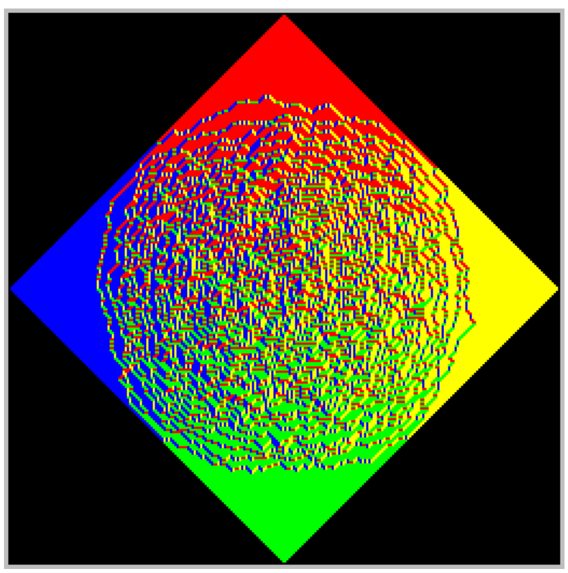

Figure 3: A tiling of a large region by dominoes.

The particular tiling shown in Figure 1 has a special property: the red and green hexagons come in matched pairs that belong to the same tile. I did not choose this particular tiling of R_4 with the pairing property in mind; this same pairing occurs in *every* tiling of R_4 by stones and bones. Any tile that would separate a red hexagon from its green partner is systematically suppressed—a subtle effect caused by the precise shape of the boundary. Such

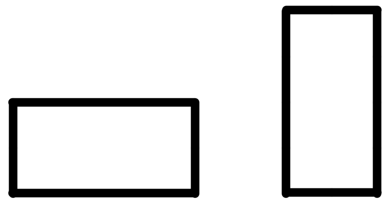

Figure 4: Dominoes.

tile suppression is a relatively recent discovery in the study of tilings, with one account given in article (1) listed below. I recently identified a simpler and more versatile explanation that uses height functions, a powerful tool in the analysis of tilings that I pioneered (see article (2)). Extending the theory of height functions into higher dimensions, as required for stones and bones, will open the door to deeper understanding of how boundary conditions shape emergent structure in polymer models. Although these systems, as models of real-world objects, are unrealistic, the properties they exhibit mirror phenomena studied in statistical physics, and my results may provide insights that will interest mathematical physicists.

Thanks to the tile suppression phenomenon, the circular boundary between order and disorder for stones-and-bones tilings in the R_n regions emerges as a consequence of an analogous result about domino tilings that I discovered with collaborators years ago. This discovery, now the subject of a Mathologer video[1], set off a burst of research at the boundary between combinatorics and probability theory. Many contributors to this wave of progress were undergraduate research assistants from the Boston area, several of whom now hold professorships at institutions such as Harvard, Cornell, the University of Michigan, and the University of Minnesota.

Three decades ago, higher-dimensional height functions were largely beyond reach; today, the field is ready for real progress. My expertise in dimers—recognized by a keynote invitation at the Dimers 2023 conference—positions me to extend these methods to polymers. I also bring decades of experience leading undergraduate research groups at MIT, Harvard, Wisconsin, Tufts, and UMass Lowell, ensuring that the project advances mathematical knowledge while continuing my long record of cultivating future mathematical researchers.

[1] "The Arctic Circle Theorem", https://www.youtube.com/watch?v=Yy7Q8IWNfHM

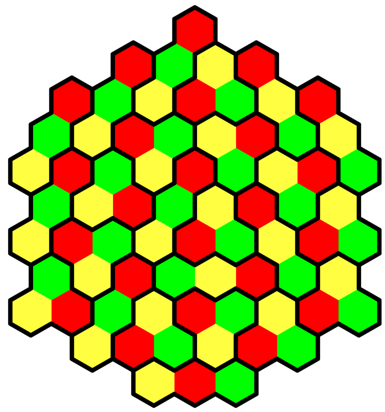

Figure 5: A tiling of a benzel of order 9.

I'm currently studying stones-and-bones tilings with seven collaborators, ranging in age from their 20s to their 50s, in the U.S., Canada, and Brazil. One goal is to understand tilings of regions called benzels. An example is shown in Figure 5. Experiments suggest that in random tilings of large benzels, three of the six corners display order. The shape of the boundary seems to make its influence felt deep into the interior of the benzel. Does this polymer model exhibit a sharp spatial transition between order and disorder, and if so, where does it lie? What curve replaces the circle seen above? Judging from my experience, I believe that smart and hard-working undergraduates will unearth several of the key ideas, and that those students need not be the ones who initially seem brilliant. One of the best ideas any of my students ever had—"condensation"—came from a quiet student who for months said very little in our group meetings. While at the Institute, I will look for more students like that. That is, I'll make an effort to identify students who love math but don't yet realize that they have the capacity to do math research, and to give them the kind of guidance that draws inspiration from my former student Federico Ardila's approach to mentorship, and from the motto "People Over Math" that he and Pamela Harris have championed.

While at Radcliffe, I'll collaborate with Prof. Lauren Williams (whom I mentored when she was a Harvard undergraduate), Prof. Noam Elkies (a past collaborator on domino tilings), Colin Defant (a Benjamin Peirce Fellow and current collaborator), and Hanna Mularczyk (an MIT graduate student and current collaborator), as well as four to six Harvard undergraduates.

I'd also be eager to collaborate with non-mathematicians, including other Radcliffe Fellows. During a 2012 visiting professorship at Berkeley, I worked with playwrights, showing how the interplay of order and randomness in tilings can serve as a metaphor for life, with competing forms of order vying for dominance. The resulting short plays, performed at the Berkeley Repertory Theatre, reflected these ideas in moving ways. That year, I also curated an art exhibit at the Mathematical Sciences Research Institute[2] showcasing mathematical images with both scientific and artistic resonance—something I'd be glad to do again at Radcliffe. More recently, I've pursued collaborative projects with math educator and artist Ben Orlin[3] exploring the playful side of mathematics.

I am selecting three recent articles as work samples.

1. *Tilings of Benzels via Generalized Compression* (2024), by Colin Defant, Leigh Foster, Rupert Li, James Propp, and Benjamin Young, published in SIAM Journal on Discrete Mathematics, exemplifies the collaborative spirit of much of my work and highlights my commitment to mentoring young scholars such as Rupert Li and Leigh Foster.

2. *Lattice Structure for Orientations of Graphs* (2025), to be published this year in the Electronic Journal of Combinatorics, is a bridge between my earliest research and my current projects.

3. *The Genius Box* (2021), published in the Journal of Humanistic Mathematics, builds on Moon Duchin's essay *The Sexual Politics of Genius*. Though not a technical work, my essay reflects the humanistic side of my interests and suggests the kind of contributions I might bring to the Institute's lunchtime conversations.

Over the past seven years, I've been active in mathematical outreach. My blog[4] attracts several thousand readers each month, and I'm currently writing a book on the concept of Number for a broad, non-academic readership. One aim of this book is to counter the Eurocentric bias that mars much popular writing on mathematics, by showing that the subject is—and always has been—a global human endeavor.

[2] "Wild Beauty: Postcards from Mathematical Worlds," https://www.slmath.org/workshops/19584

[3] Homepage at https://mathwithbaddrawings.com/

[4] Mathematical Enchantments, http://mathenchant.org