New Family of Somos-like Recurrences

Paul Heideman, Emilie Hogan

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1 Introduction

Quadratic recurrences that generate only integers, given that the first finitely many terms are 1, are an incredible phenomena of mathematics. Michael Somos, through investigations of elliptic theta functions, discovered one such quadratic recurrence that began a new wave of research into the underlying meaning of this phenomenon [4]. After Somos' discovery that the recurrence

$$a_n = (a_{n-1}a_{n-5} + a_{n-2}a_{n-4} + a_{n-3}^2)/a_{n-6}$$

generated integers when given initial conditions $a_i = 1$ for $0 \leq i \leq 5$, a surge of research began on related sequences. Most importantly, for the goals of our problem, was the sequence discovered by Dana Scott defined by the recurrence

$$a_n = (a_{n-1}a_{n-3} + a_{n-2})/a_{n-4}$$

with initial conditions $a_i = 1$ for $0 \leq i \leq 3$.

The above two sequences possess an even stronger property than integrality. If the initial conditions are given as $a_m = x_m$ for $0 \leq m \leq 5$ and $0 \leq m \leq 3$ respectively, these sequences produce Laurent polynomials. The set of Laurent polynomials are defined [6] as

$$\mathbb{R}\left[t, t^{-1}\right] = \left\{ \sum a_i t^i \mid i \in \mathbb{Z}, a_i \neq 0 \text{ for finitely many } i \in \mathbb{Z}\right\}$$

We will define Laurent polynomials in more than one variable in the same manner as above. That is, the set of Laurent polynomials in the variables $\{x, y, z\}$ is defined to be

$$\mathbb{R}\left[x, x^{-1}, y, y^{-1}, z, z^{-1}\right] = \left\{ \sum a_i x^j y^k z^l \mid i, j, k, l \in \mathbb{Z}, a_i \neq 0 \text{ for finitely many } i \in \mathbb{Z}\right\}$$

This Laurent property is discussed in Fomin and Zelevinsky's article "The Laurent Phenomenon" [3]. This property has been used to extract combinatorial information from sequences.

The work done by Dana Scott was important to our discovery since it gave us the idea to probe for sequences defined by a mixture of quadratic and linear terms, rather than just the Somos-like quadratic recurrences. Additionally, work done by Reid Barton [1] on a similar sequence resulted in a combinatorial interpretation, which led us to believe that a similar interpretation of our sequences exists.

2 Statement of Problem

While looking at the family of quadratic recurrences given by

$$a_n a_{n-k} = a_{n-1}a_{n-k+1} + a_{n-(k-1)/2} + a_{n-(k+1)/2} \tag{1}$$

with initial conditions $a_m = 1$ for $m < k$ we noticed that they gave integers for any given odd value of k we use.

For instance, when $k = 3$ the recurrence becomes

$$a_n a_{n-3} = a_{n-1}a_{n-2} + a_{n-1} + a_{n-2} \tag{2}$$

generating the sequence $\{a\} = \{1, 1, 1, 3, 7, 31, 85...\}$.

We have verified that this sequence generates integers for the first several thousand terms and for values of k in the hundreds. Our eventual goal will be to find a combinatorial interpretation (e.g. counting perfect matchings of certain graphs) and a proof of integrality, using methods analogous to those laid out by Reid Barton. In addition to a combinatorial interpretation and proof of integrality we would like to prove that when the initial conditions are $a_m = x_m$ for $m < k$ the recurrence gives Laurent polynomials.

3 First Conjecture

One of the ways to prove that a quadratic recurrence gives integers is to show that the sequence also satisfies some linear recurrence relation with integer coefficients.

Conjecture 1 *If the sequence $\{a_n\}$ is given by the quadratic recurrence (1), then it also satisfies the linear recurrence given by*

$$a_n = \left[\frac{1}{2}(k^2 + 1) + 3k\right](a_{n-k+1} - a_{n-2k+2}) + a_{n-3k+3} \tag{3}$$

for all $n \geq 0$, where k is taken to be the value used in creating $\{a_n\}$ from (1).

It is also worth noticing that proving the converse of this statement (i.e. that the linear recurrence (3) satisfies the quadratic recurrence) is equivalent to proving the statement itself. The linear recurrence (3) has been verified to coincide with the quadratic recurrence for the first 10,000 values in the sequence where k is as large as 150.

4 Inductive step of correspondence between quadratic and linear recurrences

Using methods analogous to those used by Hal Canary [2] to prove integrality of the Dana Scott recurrence, we prove the inductive step of conjecture (1).

Define the sequence $\{a_n\}$ recursively for all $n \geq 0$ **is this bound on n right???**:

$$a_n = \left[\frac{1}{2}(k^2 + 1) + 3k\right](a_{n-k+1} - a_{n-2k+2}) + a_{n-3k+3} \tag{4}$$

For simplicity, let the term $\frac{1}{2}(k^2+1)+3k$ be called $A(k)$.

Let

$$\phi(n) = a_n a_{n-k} - a_{n-1}a_{n-k+1} - a_{n-(k-1)/2} - a_{n-(k+1)/2} \tag{5}$$

We wish to prove by induction that $\phi(n) = 0$ for all n.

We make the strong induction assumption that $\phi(m) = 0$ for $m < n$. This gives:

$$\begin{aligned}
\phi(n-1) &= a_{n-1}a_{n-k-1} - a_{n-2}a_{n-k} - a_{n-(k-1)/2-1} - a_{n-(k+1)/2-1} = 0 \\
\phi(n-2) &= a_{n-2}a_{n-k-2} - a_{n-3}a_{n-k-2} - a_{n-(k-1)/2-2} - a_{n-(k+1)/2-2} = 0 \\
&\vdots \\
\phi(n-k+1) &= a_{n-k+1}a_{n-2k+1} - a_{n-k}a_{n-2k+2} - a_{n-(3k-3)/2} - a_{n-(3k-1)/2} = 0
\end{aligned}$$

We need to show that $\phi(n) = 0$ given the above conditions. Now, compute $\phi(n)$ by substituting for a_n, a_{n-1}, $a_{n-(k-1)/2}$, and $a_{n-(k+1)/2}$ from the definition of $\{a\}$.

$$\begin{aligned}
a_n &= A(k)a_{n-k+1} - A(k)a_{n-2k+2} + a_{n-3k+3} \\
a_{n-1} &= A(k)a_{n-k} - A(k)a_{n-2k+1} + a_{n-3k+2} \\
a_{n-(k-1)/2} &= A(k)a_{n-(3k-3)/2} - A(k)a_{n-(5k-5)/2} + a_{n-(7k-7)/2} \\
a_{n-(k+1)/2} &= A(k)a_{n-(3k-1)/2} - A(k)a_{n-(5k-3)/2} + a_{n-(7k-5)/2}
\end{aligned}$$

With these substitutions, $\phi(n)$ simplifies to

$$\begin{aligned}
\phi(n) = \quad & (A(k)a_{n-k+1} - A(k)a_{n-2k+2} + a_{n-3k+3})a_{n-k} - \\
& (A(k)a_{n-k} - A(k)a_{n-2k+1} + a_{n-3k+2})a_{n-k+1} - \\
& (A(k)a_{n-(3k-3)/2} - A(k)a_{n-(5k-5)/2} + a_{n-(7k-7)/2}) - \\
& (A(k)a_{n-(3k-1)/2} - A(k)a_{n-(5k-3)/2} + a_{n-(7k-5)/2})
\end{aligned}$$

Expand and cancel and you get

$$\begin{aligned}
\phi(n) = \quad & -A(k)a_{n-2k+2}a_{n-k} + A(k)a_{n-2k+1}a_{n-k+1} - A(k)a_{n-(3k-3)/2} - A(k)a_{n-(3k-1)/2} + \\
& a_{n-k}a_{n-3k+3} - a_{n-k+1}a_{n-3k+2} + A(k)a_{n-(5k-5)/2} + A(k)a_{n-(5k-3)/2} - \\
& a_{n-(7k-7)/2} - a_{n-(7k-5)/2}
\end{aligned}$$

You can see that

$$-A(k)a_{n-2k+2}a_{n-k} + A(k)a_{n-2k+1}a_{n-k+1} - A(k)a_{n-(3k-3)/2} - A(k)a_{n-(3k-1)/2} = -A(k)\phi(n-k+1)$$

which equals 0 by the induction hypothesis.

Thus

$$\phi(n) = a_{n-k}a_{n-3k+3} - a_{n-k+1}a_{n-3k+2} + A(k)a_{n-(5k-5)/2} + A(k)a_{n-(5k-3)/2} - a_{n-(7k-7)/2} - a_{n-(7k-5)/2}$$

Substitute for a_{n-k+1} and a_{n-k} from the definition of $\{a\}$.

$$\begin{aligned}
a_{n-k+1} &= A(k)a_{n-2k+2} - A(k)a_{n-3k+3} + a_{n-4k+4} \\
a_{n-k} &= A(k)a_{n-2k+1} - A(k)a_{n-3k+2} + a_{n-4k+3}
\end{aligned}$$

After these are substituted in, expand and cancel again to obtain

$$\begin{aligned}\phi(n) &= A(k)a_{n-2k+1}a_{n-3k+3} - A(k)a_{n-2k+2}a_{n-3k+2} - A(k)a_{n-(5k-5)/2} - A(k)a_{n-(5k-3)/2} \\ &= -A(k)\phi(n-2k+2) + \phi(n-3k+3) \\ &= 0\end{aligned}$$

Note that this proof was only the inductive step. This is because the base case only works for specific values of k. While it is easy to show the base case for small finite values of k, it is a rather difficult task to prove the base case for general k because the number of terms in the base case grows unbounded and linearly as k increases. However, for small finite values of k, proving the base case requires nothing more that a few lines and minimal cranial exercise. For example, in the $k = 3$ case the statement of the proof and base case go as follows:

Define the sequence $\{a\}$ recursively:

$$a_n = 14a_{n-2} - 14a_{n-4} + a_{n-6} \tag{6}$$

With the initial conditions $(a_1, ..., a_6) = (1, 1, 1, 3, 7, 31)$. Using the above definition for a_n, find the next term in the sequence. By simple arithmetic we find that $a_7 = 14 \cdot 7 - 14 \cdot 1 + 1 = 85$. In order to prove the base case, we need to show that the sequence $\{a_4, a_5, a_6, a_7\}$ satisfies

$$a_n a_{n-3} - a_{n-1}a_{n-2} - a_{n-1} - a_{n-2} = 0$$

where $n = 7$. It can be verified by simple arithmetic $85 \cdot 3 - 31 \cdot 7 - 31 - 7 = 0$.

Note that if we can prove that this type of proof of the base case is valid for any odd integer value of k then we have the generalized base case and thus the entire induction proof.

5 Generalization of Quadratic Recurrences

While looking at the sequences mentioned above, we came across a more broad definition of these sequences. With this generalization, it happens that the same relationship works for both k even and k odd.

$$b_n b_{n-k} = b_{n-i}b_{n-k+i} + b_{n-j} + b_{n-k+j} \tag{7}$$

with the conditions that $i < k - i < k$, $j < k - j < k$, and $b_r = 1$ for all $r < k$.

With regards to the equation (7) we have come up with integrality conjectures. They are broken up into the case where k is even and the case where k is odd.

Conjecture 2 *Consider the general form of the quadratic recurrence (7) with initial terms $b_i = 1$ for $0 \le i \le k$.*
If k is even, then
(1a) If i is odd, then $j = \frac{k}{2}$ defines a recurrence that generates only integers.
(2a) If i is even, then $j = \frac{i}{2}$, $j = \frac{k}{2}$, and $j = \frac{k-1}{2}$ define recurrences that generate only integers.
If k is odd, then
(1b) If i is odd then $j = \frac{k-1}{2}$ defines a recurrence that generates only integers.
(2b) If i is even then $j = \frac{k}{2}$ defines a recurrence that generates only integers.

Furthermore, all other values of j do not define a recurrence that gives integers exclusively.

For example, the recurrence (1) is a specific example of the generalization (7) where k is odd, and $i = 1$.

6 Lifting the recurrences to 2-dimensional space

These recurrences, however, are expected to satisfy even stronger conditions than integrality. We expect, and have empirically verified for numerous cases, that if we instead take the initial conditions of the a_n family to be formal variables, all terms will be Laurent polynomials of those formal variables. However, these polynomials will have increasingly large coefficients. For the purposes of extracting combinatorics, it is preferable to modify these recurrences so that solutions are Laurent polynomials with all coefficients of 1 (which we will call *faithful* polynomials) so that each term counts some object (e.g. a perfect matching).

Conjecture 3 *For all conjectured families of recurrences in section 5 except for the cases where k is even and $j = \frac{k}{2}$, the two dimensional recurrence*

$$a_{n,k}a_{n-k,k} = a_{n-i,k+2}a_{n-k+i,k-2} + a_{n-j,k+1} + a_{n-k+j,k-1}$$

with initial terms $a_{i,j} = x_{i,j}$ for all $i < 0$ will generate faithful Laurent polynomials in $x_{i,j}$.

Note that when k is even and $j = \frac{k}{2}$, we still obtain a Laurent polynomial with this method, but by combining equal terms one can see immediately that it is not faithful.

Though it is not entirely clear what this lifting process will give us, it has been a means for finding combinatorics in other similar sequences such as the Reid Barton and Dana Scott recurrences [1].

7 Recovering faithful genuine polynomials from faithful Laurent Polynomials

To recover the combinatorics of these sequences, it is beneficial to recover faithful genuine polynomials from the lifted faithful Laurent Polynomials described in the previous section. This will allow for each term to be viewed as an object (such as a perfect matching of a graph) and each factor of that term to be viewed as an edge "weight" of that graph. For example, consider the specific sequence lifted as described in section 6.

$$a_{n,k}a_{n,k-3} = a_{n-1,k+2}a_{n-2,k-2} + a_{n-1,k-1} + a_{n-2,k+1}$$

We can "tilt" this recurrence (analogous to the tilting trick used by Fomin and Zelevinsky [3]) and use Maple to write the program

```
B:= proc(i,j) option remember;
if i+j<3 then x(i,j)
else simplify((B(i-2,j)*B(i,j-1)
+B(i-1,j)+B(i-1,j-1))/B(i-2,j-1));
fi;end;
```

If you replace the `x(i,j)` with `1` it is clear that this program just outputs the sequence in integers. However, there should be ways of putting in coefficients of the `B(i,j)` terms so that what you get out of the program are faithful Laurent polynomials. We have found two such ways of inserting coefficients. The first will give rise to non-faithful Laurent polynomials, the second gives non-faithful genuine polynomials.

```
B:= proc(i,j) option remember;
if i+j<3 then 1
else simplify((u(i-1)*v(j-1)*u(i+1)*v(j)*B(i-2,j)*B(i,j-1)+
u(i)*v(j-1)*u(i)*v(j)*(B(i-1,j)+B(i-1,j-1)))/
(u(i+1)*v(j-1)*u(i-1)*v(j)*B(i-2,j-1)));fi;end;

B:= proc(i,j) option remember;
if i+j<3 then 1
else simplify((B(i-2,j)*B(i,j-1)
+u(i)*v(j-1)*B(i-1,j)+
u(i)*v(j)*B(i-1,j-1))/B(i-2,j-1));fi;end;
```

It is not yet known what these `u(i)` and `v(j)` coefficients mean. However, in previous examples (e.g. Musiker and Propp) [5] where this technique was used, the coefficients have corresponded to edge weights of certain graphs. The new sequence corresponded to perfect matchings of these graphs.

8 Conclusion

The new family of sequences that we found are interesting in the sense that it is expected follow the Laurent phenomenon. It is our hope that we find combinatorial objects that our family of sequences counts, possibly even a sequence of combinatorial objects that follow a nice rule according to the parameters of successive individual sequences.

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