Spaces of tilings

a talk in honor of Nicolau Saldanha

Jim Propp, UMass Lowell

presented on the 8th day
of the 8th month
of the year 8×253

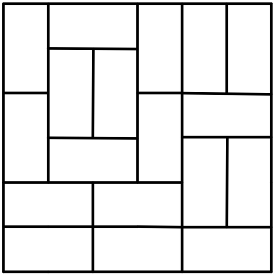
All slides available at

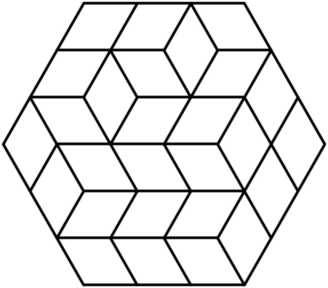
http://faculty.uml.edu/jpropp/saldanha60.pdf

Thank you, Nicolau, especially for:

Spaces of domino tilings by N. C. Saldanha, C. Tomei, M. A. Casarin, Jr., and D. Romualdo, Discrete Comput Geom 14: 207-233 (1995)

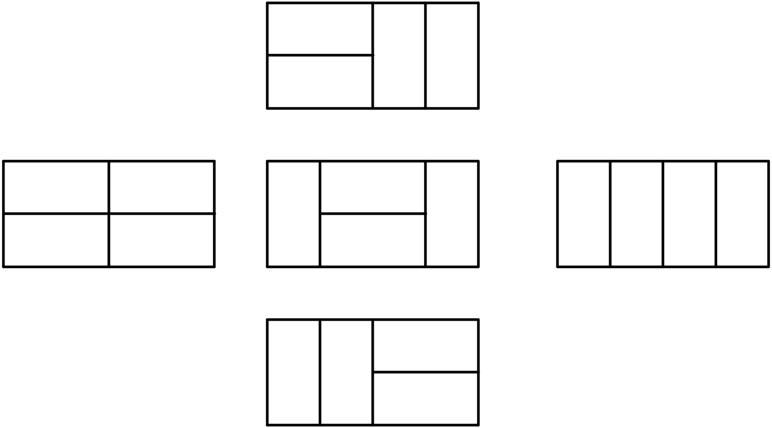
An overview of domino and lozenge tilings by Nicolau C. Saldanha and Carlos Tomei (1998)





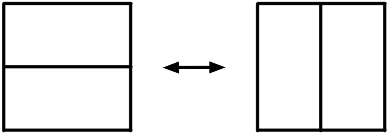
What is a space of tilings?

We are interested in studying T, the set of all possible tilings of a fixed region.



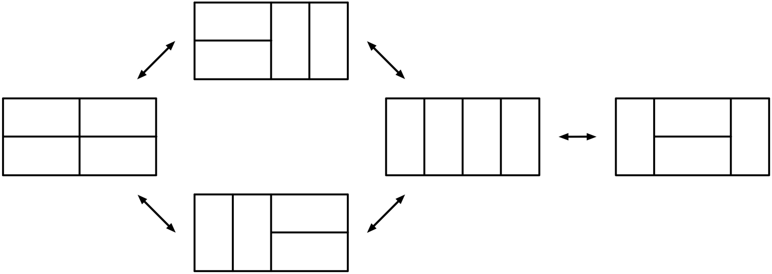
What is a space of tilings?

Given a tiling, we perform a *flip* by lifting two dominoes and placing them back in a different position: clearly, the two dominoes must form a square of side 2.



What is a space of tilings?

Two tilings are *adjacent* in T if we move from one to the other by a flip. Turn T into a graph by joining adjacent tilings by edges and define connected components of T and distance between tilings in the usual way.



What is a space of tilings?

Our techniques provide us with a fair understanding of the combinatorial, topological, and metric structure of T

...

we describe in Theorem 3.2 a simple formula for the distance between tilings and a characterization of shortest routes between points.

...

$$d(t_1, t_2) = \frac{1}{4} \sum_{p \in A^*} |\theta_1(p) - \theta_2(p)|.$$

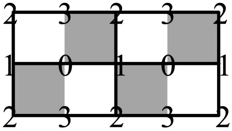
Here t_1 and t_2 are tilings of a simply-connected region, $d(t_1, t_2)$ is the flip-distance between them, A^* is the set of vertices of the square cells that underlie both tilings, and θ_1 and θ_2 are **height functions** associated with t_1 and t_2.

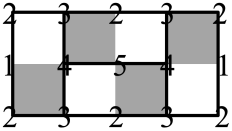
Height functions

Height goes up by 1 (resp. down by 1) when you travel along a tile-edge with a shaded (resp. unshaded) square to your left.

The height function determines the tiling: an edge with endpoints p, q is a tile-edge if and only if $|\theta(p) - \theta(q)| = 1$.

Check the formula for $d(t_1, t_2)$:

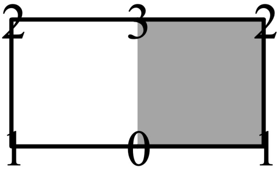




$$\frac{1}{4}\big(|0 - 4| \ + \ |1 - 5| \ + \ |0 - 4|\big) \ = \ 3$$

Height functions

Notice that the height function associated with a domino tiling is locally consistent because each tile is color-balanced (going around a tile causes the height to change by $3 - 3 = 0$) and it's globally consistent because of local consistency combined with the simply-connectedness of the region.

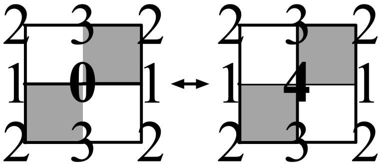


Height functions

Also notice that if t_1 and t_2 are two domino tilings that are related by a single flip,

$$\frac{1}{4}\sum_{p\in A^*}|\theta_1(p)-\theta_2(p)|=1$$

because $\theta_1(p)-\theta_2(p)=\pm 4$ if p is the center of the 2-by-2 square that has been flipped and $\theta_1(p)-\theta_2(p)=0$ otherwise.



Height functions

This shows that for general t_1 and t_2,

$$d(t_1, t_2) \geq \frac{1}{4} \sum_{p \in A^*} |\theta_1(p) - \theta_2(p)|$$

by the triangle inequality. The real work is showing that equality holds.

Once that work is done, we obtain an interpretation of the individual terms of the sum: $|\theta_1(p) - \theta_2(p)|$ equals the number of flips that get performed on the 2-by-2 square centered at p, along every geodesic in T from t_1 to t_2.

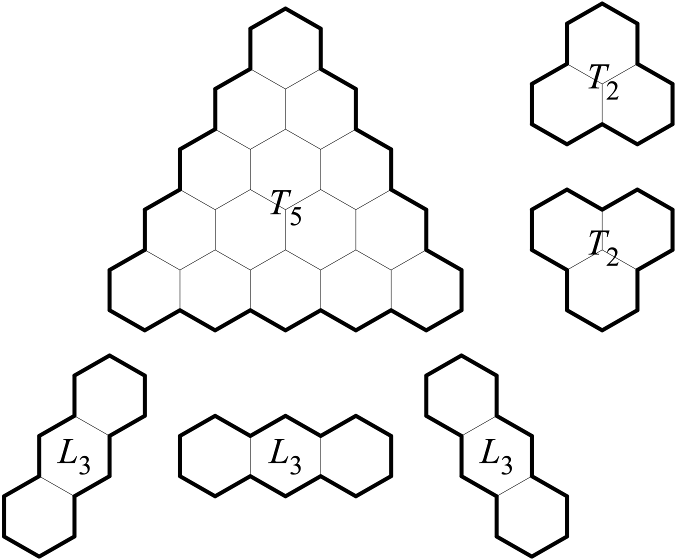
Background

Conway and Lagarias [1] studied the problem of tiling a subset of $\mathbb{R}^2$ with a given set of tiles, by group-theoretical techniques. Thurston [10] adapted these techniques to study domino tilings, producing a necessary and sufficient condition for a simply connected region of the plane to be tileable by dominoes.

[1] J. H. Conway and J. C. Lagarias, Tilings with polyominoes and combinatorial group theory, J. Combin. Theory Ser. A, 53, 183–208 (1990).

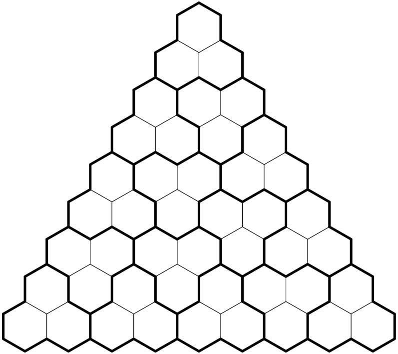
[10] W. P. Thurston, Conway's tiling groups, Amer. Math. Monthly, 97(8) 757–773 (1990).

Conway and Lagarias



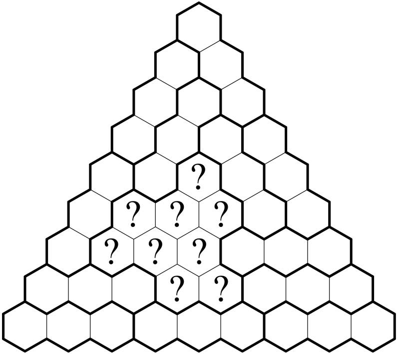
Conway and Lagarias

Theorem (C & L):
(1) T_n can be tied by T_2's precisely when n is 0, 2, 9, or 11 (mod 12).



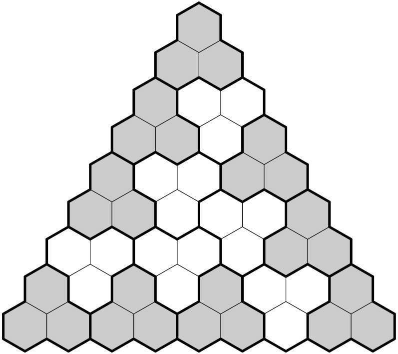
Conway and Lagarias

Theorem (C & L):
(2) T_n can be tied by L_3's precisely when n is 0 (i.e., never, outside of the trivial case).



Conway and Lagarias

They proved their results by using a powerful lemma: given a simply-connected region in the plane, then the number of up-pointing T_2's minus the number of down-pointing T_2's is an <u>invariant</u> – that it, it is the same for all tilings of the region.



$$9 - 6 \neq 0$$

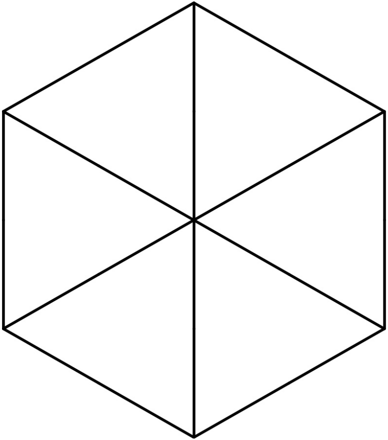
Thurston

Conway and Lagarias proved their lemma using combinatorial group theory.

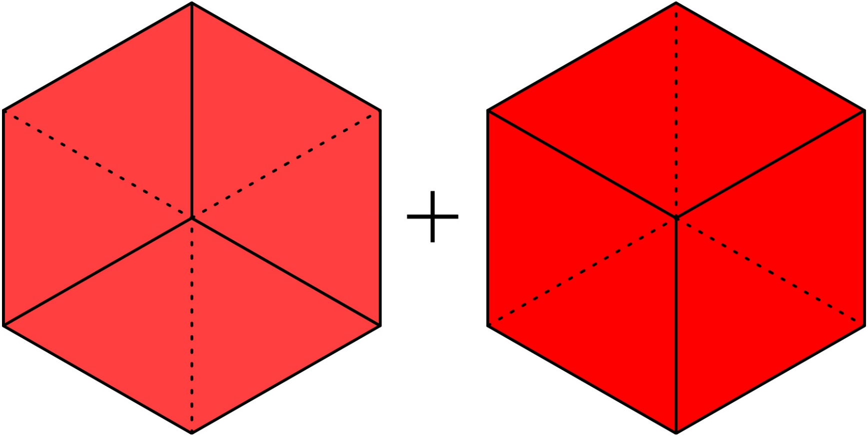
Thurston restyled the core idea in topological language: "Represent the region being tiled as a 2-complex, remove the faces associated with the cells, replace them by new faces associated with the tiles, and look at homotopies of paths in the new 2-complex."

This "gluing in disks" viewpoint shed new light on domino and lozenge tilings.

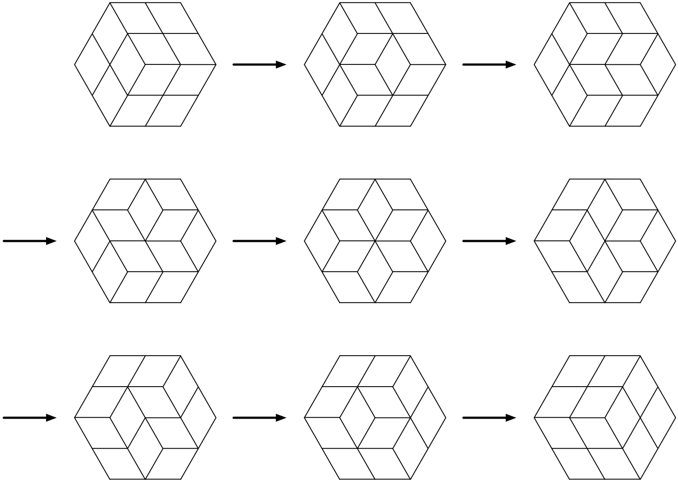
Thurston



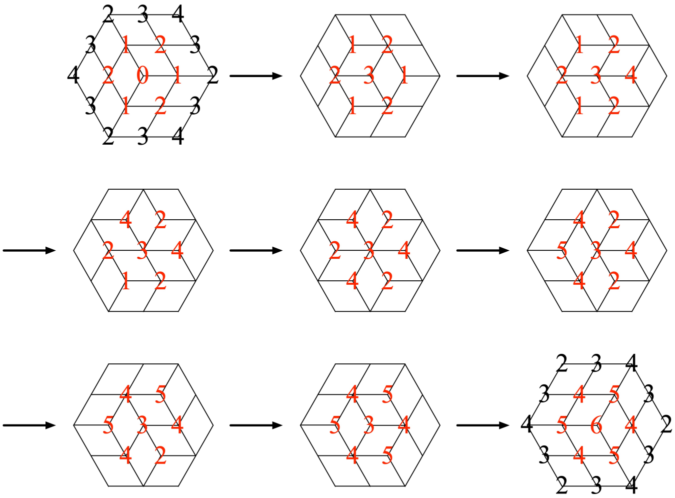
Thurston



Thurston



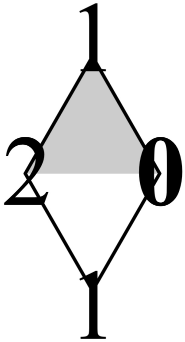
Thurston



Thurston

Height goes up (resp. down) by 1 when you travel with an upward- (resp. downward-) pointing triangle to your left.

The height function is globally consistent and determines the tiling: an edge with endpoints p, q is a tile-edge if and only if $|\theta(p) - \theta(q)| = 1$.

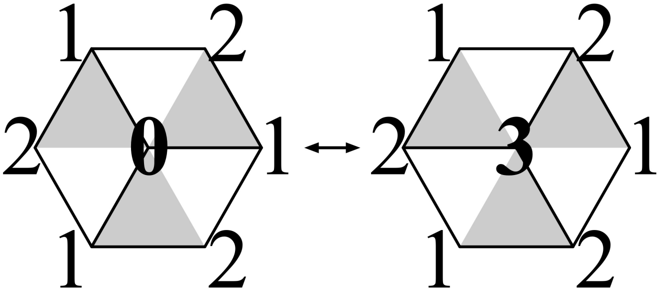


Thurston

Notice that if t_1 and t_2 are two lozenge tilings that are related by a single flip,

$$\frac{1}{3}\sum_{p\in A*}|\theta_1(p)-\theta_2(p)|=1$$

because $\theta_1(p)-\theta_2(p)=\pm 3$ if p is the center of the unit hexagon that has been flipped and $\theta_1(p)-\theta_2(p)=0$ otherwise.

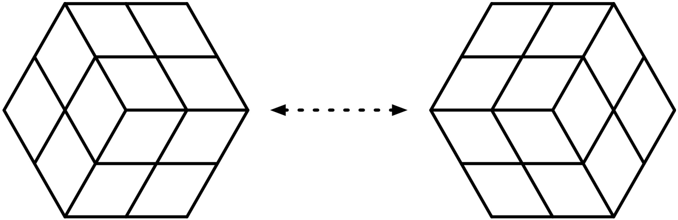


Thurston

For lozenge tilings of simply-connected regions,

$$d(t_1, t_2) = \frac{1}{3} \sum_p |\theta_1(p) - \theta_2(p)|$$

E.g., when the region being tiled is a regular hexagon of side length n, the diameter of the space of tilings is n^3.

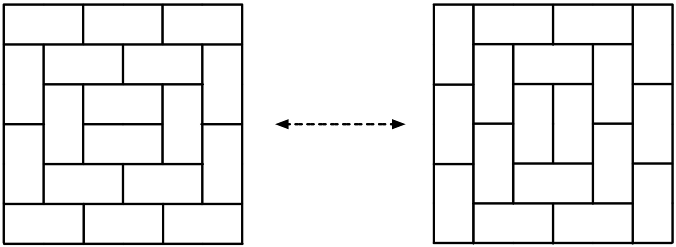


Thurston

For domino tilings of simply-connected regions,

$$d(t_1, t_2) = \frac{1}{4} \sum_p |\theta_1(p) - \theta_2(p)|$$

E.g., when the region being tiled is a square of side length $2n$, the diameter of the space of tilings is $n(2n-1)(2n+1)/3$.



Thurston

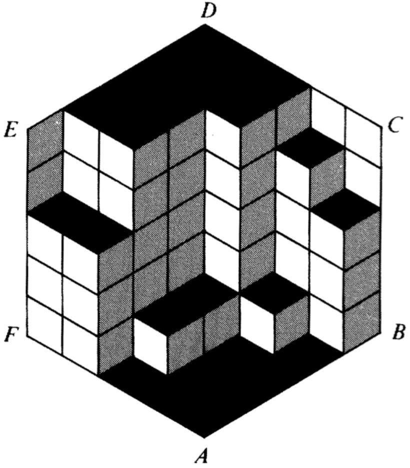
Under Thurston's approach, height functions enable one to put a distributive lattice structure on the set of tilings.

Specifically, there is a "highest tiling" and a "lowest tiling", and the distance between them is the diameter of the graph T.

This lattice structure can play a role in helping one prove theorems about the tilings of a region.

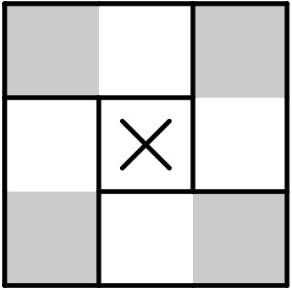
David and Tomei

See also Guy David and Carlos Tomei's The Problem of the Calissons (American Mathematical Monthly, Vol. 96 (1989), pp. 429-431).



Saldanha et al.

But what if the region we're tiling has holes?



Then the definition of height functions may not work.

Saldanha et al.

Saldanha, Tomei, Casarin, and Romualdo introduced height-sections to deal with problems like this.

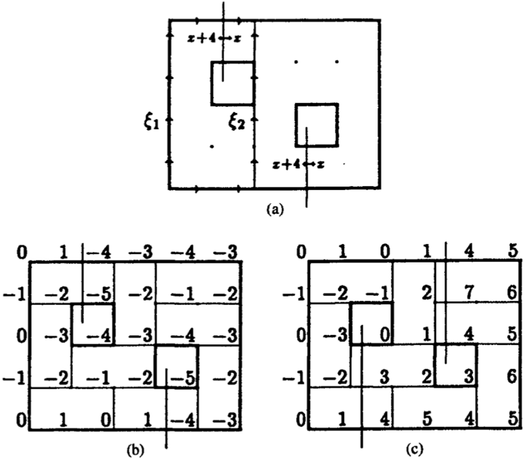
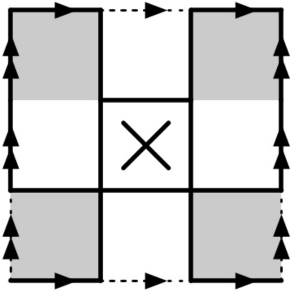


Fig. 2.2. The same height section for different cuts.

Saldanha et al.

Okay, but what if the region doesn't live in the plane but in a surface of higher genus?



Then there may not be a globally consistent checkerboard coloring of the cells.

Saldanha et al.

Saldanha, Tomei, Casarin, and Romualdo have techniques for that as well.

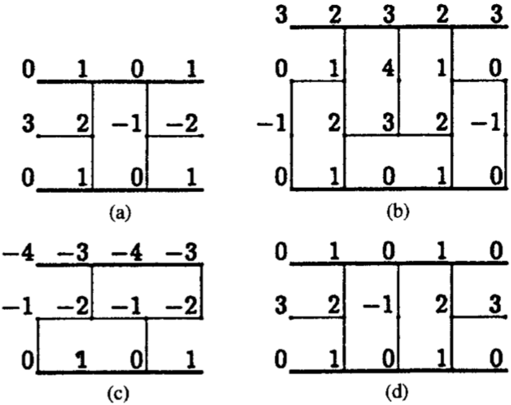


Fig. 4.3. Height sections in cylinders and Möbius bands.

Other work from the 1990s

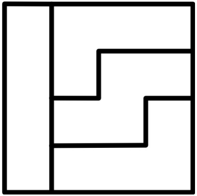
S. Kannan and D. Soroker, Tiling polygons with parallelograms, Discrete Comput. Geom., Volume 7, 175-188 (1992)

R. Kenyon, Tiling a polygon with parallelograms, Algorithmica, Volume 9, 382-397 (1993)

J. Propp, Lattice structure for orientations of graphs, in preparation (1995), to be published in 2024 or 2025. (See section 5 of that article for more of the story of height functions.)

Results from the 2000s

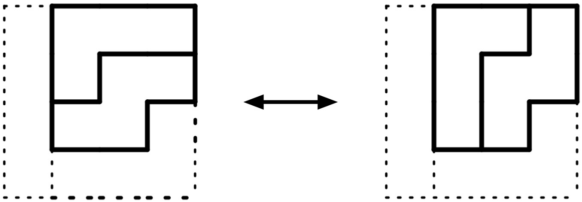
Igor Pak and others extended the Conway-Lagarias-Thurston approach to the study of ribbon tilings, e.g., 4-ribbon tilings:



See R. Muchnik and I. Pak, On tilings by ribbon tetrominoes (Journal of Combinatorial Theory, Ser. A, vol. 88 (1999), 188-193); I. Pak, Ribbon tile invariants (Trans. AMS., vol. 352 (2000), 5525-5561); and C. Moore and I. Pak, Ribbon tile invariants from the signed area (Journal of Combinatorial Theory, Ser. A, vol. 98 (2002), 1-16).

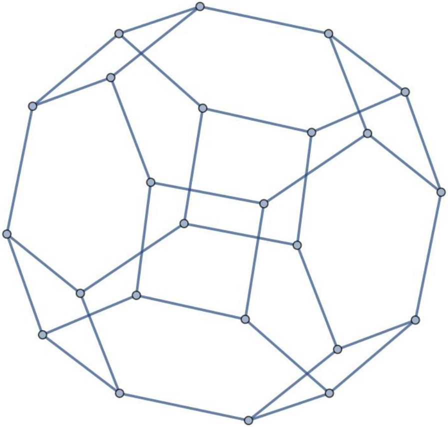
Results from the 2000s

In his article Ribbon tilings and multidimensional height functions Scott Sheffield used height functions to resolve a conjecture of Pak's, showing that for any two k-ribbon tilings t_1, t_2 of a simply-connected region, t_2 can be obtained from t_1 via a sequence of 2-flips like this:



Results from the 2000s

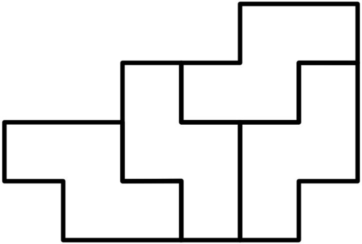
Fun fact (Cris Moore): The number of n-ribbon tilings of the n-by-n square is $n!$.



The space of tilings is the 1-skeleton of the permutahedron.

Results from the 2000s

In his 2004 doctoral thesis Geometric and algebraic properties of polyomino tilings, Michael Korn showed that the set of "skew-tetrominoes" does not have a local-move property for tilings of simply-connected regions.



That is, for every k there exist two tilings t_1, t_2 of some simply-connected plane region R such that it is impossible to turn t_1 into t_2 via moves that replace k tiles by k other tiles.

Results from the 2010s

In his 2014 paper The topology of tile invariants, Rocky Mountain Journal of Mathematics, vol. 45 (2015), 539–564) Michael Hitchman showed how some of Pak's work on ribbon tilings could be understood in terms of the second homology group of a 2-complex built from T.

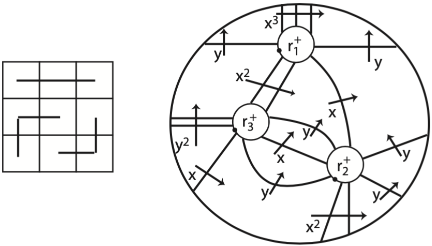


FIGURE 9. Converting a tiling of a region by the tromino set T_3 into a picture over $\mathcal{P}_3$.

Results from the 2010s

In 2015 and after, Nicolau Saldanha and others explored ways to extend Thurston's result on dominoes to three dimensions.

Short summary: if you want connectivity-under-moves, you'll need extra hypotheses and/or new kinds of moves.

See P. H. Milet and N. C. Saldanha, Flip invariance for domino tilings of three-dimensional regions with two floors (Discrete & Computational Geometry **53**, 914-940 (2015)) and assorted preprints (arXiv:1410.7693, arXiv:1411.1793, arXiv:1503.04617, and arXiv:1912.12102).

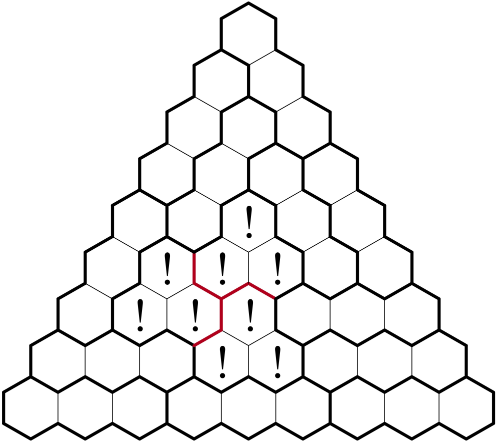
Results from the 2020s

Saldanha and others explored further extensions, some of them progressing to four dimensions and beyond.

See J. Freire, C. Klivans, P. Milet, N. Saldanha, On the connectivity of three dimensional tilings (Transactions of the American Mathematical Society **375**, 1579-1605 (2021)); N. C. Saldanha, Domino tilings of cylinders: connected components under flips and normal distribution of the twist (Electronic Journal of Combinatorics, P1.28 (2021)); and C. Klivans and N. C. Saldanha, Domino tilings and flips in dimensions 4 and higher (Algebraic Combinatorics **5** 163-185 (2022)).

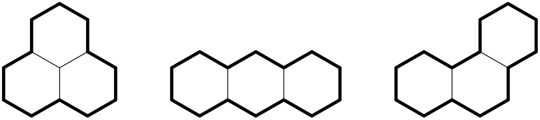
Results from this year

But what about the problem from Conway and Lagarias' original article, where a T_n region is to be tiled with T_2 tiles or L_3 tiles, and we allow both kinds of tiles?



New terminology

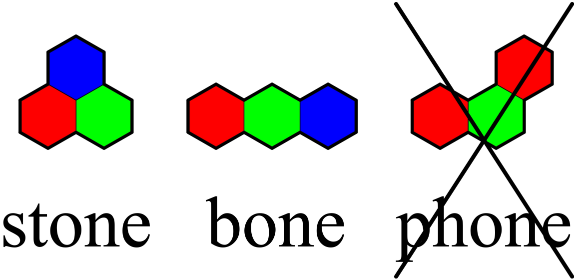
The T_2 and L_3 tiles are two of the three kinds of trihexes. In the spirit of recreational math, I've rebranded them as the stone, the bone, and the phone.



stone bone phone

New terminology

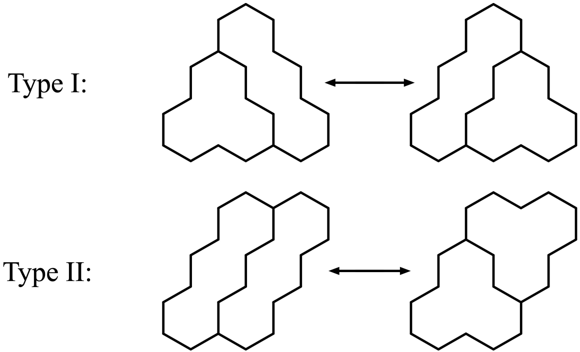
Under the natural three-coloring of the hexagonal grid, every stone and bone has a single cell of each other; not so for phones. So, let's forbid them.



We'll color the hexagonal cells blue (color 1), green (color 2), and red (color 3).

An old (?) conjecture

Conjecture: For any two stones-and-bones tilings t_1 and t_2 of a fixed simply-connected region, the following two kinds of moves (applied in succession) suffice to convert t_1 into t_2:



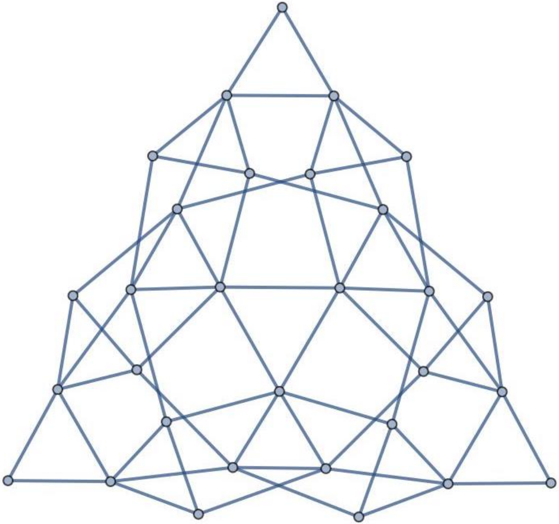
An old (?) conjecture

I'm not sure who first conjectured this, but my best guess is that it was me, circa 2000 (in emails).

In any case, Colin Defant, Leigh Foster, Rupert Li, Hanna Mularczyk, Cris Moore, Benjamin Young and I are working toward a proof. For updates, see entry for problem 19 in the Benzel Tilings Site.

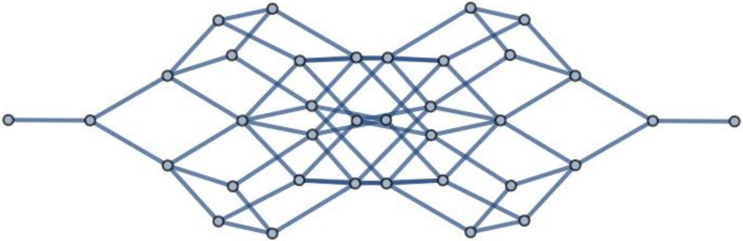
Stones-and-bones tilings of a triangle

Here is the space of stones-and-bones tilings of a triangle:

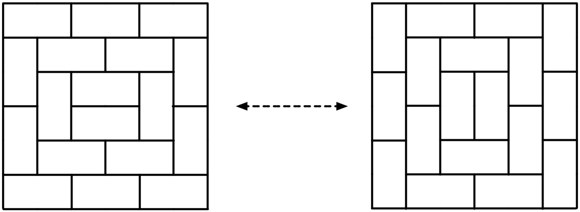


Domino tilings of a square

For comparison, here is the space of domino tilings of a square:

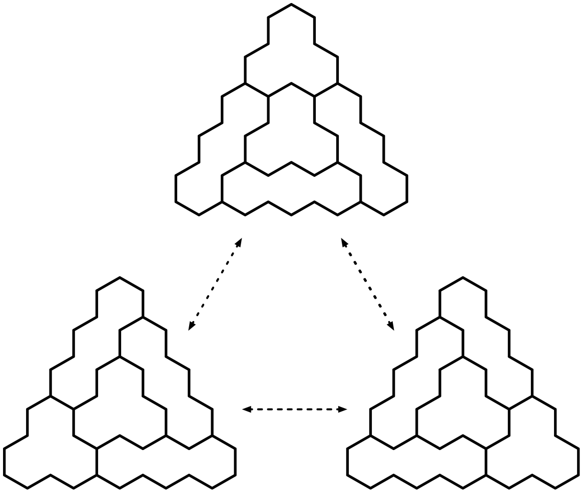


Extremal domino tilings of the square



The diameter of T is the distance between those two tilings.

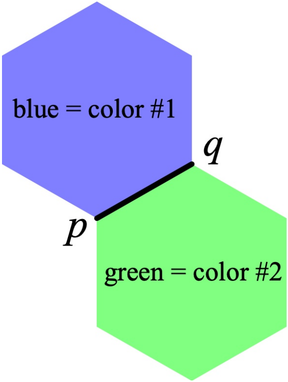
Extremal stones-and-bones tilings of the triangle



The diameter of T is the distance between any two of them.

"Height functions"

A natural kind of height is given by a triple of integers that goes up by $e_i - e_j$ when you travel along a tile-edge with a color i to your left and color j to your right (with $e_1 = (1,0,0)$ etc.).



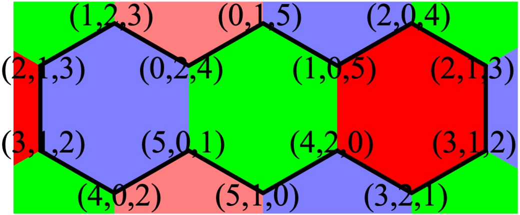
$$\theta(q) = \theta(p) + (1, -1, 0)$$

"Height functions"

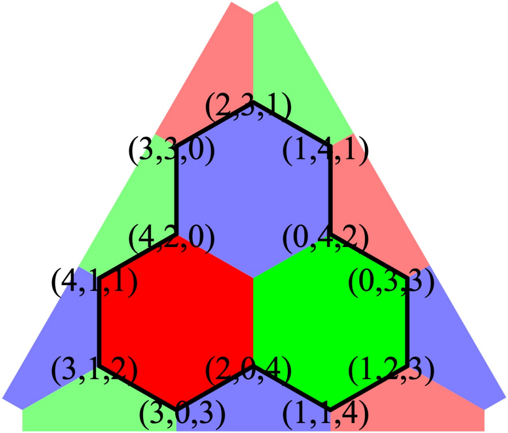
This height function is locally consistent because each tile is color-balanced, and globally consistent because of local consistency and simply-connectedness of the region.

It also retains all the information in the tiling: two vertices p, q that are adjacent in the underlying cell-complex of hexagons are joined by an edge in the tiling t if and only if the corresponding height function θ has the property that $||\theta(p) - \theta(q)||_1$ is 2, where $|| \cdot ||_1$ is the L_1-norm.

Consistency

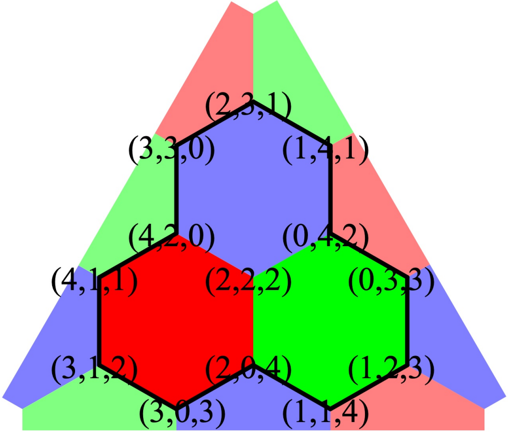


Consistency



Consistency

We assign each vertex in the interior of a stone a height equal to the average of the heights of its three neighbors.



“Height functions”

All triples (a, b, c) that occur as heights of vertices have the same value of $a + b + c$, so our three-dimensional heights are really two-dimensional.

It’s conceptually convenient to take $a + b + c = 0$ but it’s psychologically preferable to choose heights so that all integers that occur are non-negative (as in the figures).

It can be easily checked that if t_1 and t_2 are related by a Type I or Type II move, $\sum_p ||\theta_1(p) - \theta_2(p)||_1 = 36$.

So for general t_1 and t_2, by the triangle inequality, the (conceivably infinite!) moves-distance between the two tilings is bounded below by the sum

$$\frac{1}{36} \sum_p ||\theta_1(p) - \theta_2(p)||_1$$

A new, stronger conjecture

Conjecture (2024): For stones-and-bones tilings t_1, t_2 of a simply-connected region, the moves-distance $d(t_1, t_2)$ is given by the formula

$$d(t_1, t_2) = \frac{1}{36} \sum_{p} ||\theta_1(p) - \theta_2(p)||_1$$

That is, the height-function bound on moves-distance is tight.

Ample experimental evidence supports the conjecture.

The tileability problem

Thurston used height functions to give a linear time algorithm for determining whether a region could or could not be tiled by dominoes or lozenges (here "linear" means "linear in area").

This approach was extended further in the 1990s; see T. Chaboud, Domino tilings in planar graphs with regular and bipartite dual (Theoret. Comput. Sci. 159 (1996), 137-142) and J. C. Fournier, Tiling pictures of the plane with dominoes (Discrete Math. 165/166 (1997), 313-320).

The tileability problem

A key lemma is that in the lowest tiling of a (tileable) region R, all the vertices with maximal height are on the boundary of R. This allows one to construct the lowest tiling of R from the outside in. If the algorithm fails on a region R, that failure provides a proof that no tiling of R exists.
An alternative algorithm comes from the approach Saldanha et al. described in 1995:

> It is clear from the proof above that we know which flips to perform in order to get closer to a tiling t_2 starting from a tiling t_1: simply compute both height sections and look for local minima of t_1 below t_2, or local maxima of t_1 above t_2. In this sense there is a local characterization of the shortest paths in the graph $T(A)$.

(In the case of domino tilings an even faster algorithm was devised by Pak, Sheffer, and Tassy (Fast Domino Tileability, Discrete Comput. Geom. **56** (2016), 377-394); the running time is only slightly greater than linear in the *perimeter*.)

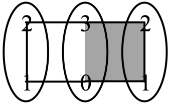
The tileability problem

Likewise, in his paper on ribbon tilings Scott Sheffield used height functions to give a linear time algorithm for determining when a simply-connected region in the plane can be tiled by k-ribbon tiles.

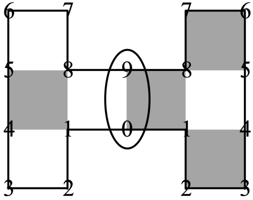
Can we do something similar for stones-and-bones tilings?

The tileability problem

One thing that is clear is that for stones-and-bones tilings we can sometimes (always?) use height functions to get succinct certificates of non-tileability, just as we do for domino tilings.

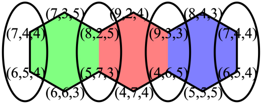


Difference = 1 or 3

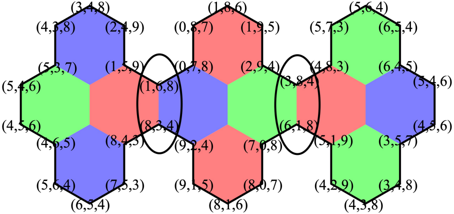


Difference = 9

The tileability problem



Distance = 2 or 10



Distance = 14

Thanks!

Thanks to the organizers of this conference for inviting (and supporting me), and thanks to Nicolau for his work on tilings!

All slides available at

http://faculty.uml.edu/jpropp/saldanha60.pdf