1 Introduction

The **Aztec diamond of order** n is the union of those lattice squares $[a, a+1] \times [b, b+1] \subset \mathbf{R}^2$ $(a, b \in \mathbf{Z})$ that lie completely inside the tilted square $\{(x, y) : |x| + |y| \leq n + 1\}$. (Figure 1 shows the Aztec diamond of order 3.) A **domino** is a closed 1×2 or 2×1 rectangle in $\mathbf{R}^2$ with corners in $\mathbf{Z}^2$, and a **tiling** of a region R by dominoes is a set of dominoes whose interiors are disjoint and whose union is R. In this article we will show that the number of domino tilings of the Aztec diamond of order n is $2^{n(n+1)/2}$. We will furthermore obtain more refined enumerative information regarding two natural statistics of a tiling: the number of vertical tiles and the "rank" of the tiling (to be defined shortly).

Fix a tiling T of the Aztec diamond of order n. Every horizontal line $y = k$ divides the Aztec diamond into two regions of even area; it follows that the number of dominoes that straddle the line must be even. Letting k vary, we see that the total number of vertical dominoes must be even; accordingly, we define $v(T)$ as half the number of vertical tiles in T.

The most intuitively accessible definition of the rank-statistic $r(T)$ comes by way of the notion of an "elementary move", which is an operation that converts one domino-tiling of a region into another by removing two dominoes that form a 2×2 block and putting them back rotated by 90 degrees (see Figure 2). It will be shown that any domino-tiling of an Aztec diamond can be reached from any other by a sequence of such moves; we may therefore define the **rank** of the tiling T as the minimum number of moves required to reach T from the "all-horizontals" tiling (shown on the left side of Figure 2). Thus the all-horizontals tiling itself has rank 0, while the tiling shown on the right side of Figure 2 (viewed as a tiling of the order 1 Aztec diamond) has rank 1.

Let

$$\mathrm{AD}(n; x, q) = \sum_T x^{v(T)} q^{r(T)}$$

where T ranges over all domino tilings of the order-n Aztec diamond; this is a polynomial in x and q. The main result of this paper is:

THEOREM:

$$\mathrm{AD}(n; x, q) = \prod_{k=0}^{n-1} (1 + xq^{2k+1})^{n-k}.$$

As important special cases, we have

$$\begin{aligned}
\mathrm{AD}(n;q) &= \prod_{k=0}^{n-1}(1+q^{2k+1})^{n-k}, \\
\mathrm{AD}(n;x) &= (1+x)^{n(n+1)/2}, \text{ and} \\
\mathrm{AD}(n) &= 2^{n(n+1)/2},
\end{aligned}$$

where we adopt the convention that an omitted variable is set equal to 1.

We will give four ways of understanding the formula for $\mathrm{AD}(n)$. The first exploits the relationship between tilings of the Aztec diamond and the still fairly mysterious "alternating sign matrices" introduced by Mills, Robbins, and Rumsey in [**10**]. Our second proof yields the formula for $\mathrm{AD}(n)$ as a special case of a theorem on monotone triangles (combinatorial objects closely related to alternating sign matrices and introduced in [**11**]). The third proof comes from the representation theory of the general linear group. The last proof yields the more general formula for $\mathrm{AD}(n;x,q)$, and also leads to a bijection between tilings of the order-n diamond and bit-strings of length $n(n+1)/2$. We conclude by pointing out some connections between our results and the "square ice" model studied in statistical mechanics.

2 Height functions

It is not at all clear from the definition of rank given in section 1 just how one would calculate the rank of a specific tiling; for instance, it happens that the all-verticals tiling of the order-n Aztec diamond has rank $n(n+1)(2n+1)/6$ and that every other tiling has strictly smaller rank, but it is far from obvious how one would check this. Therefore, we will now give a more technical definition of the rank, and prove that it coincides with the definition given above. We use the vertex-marking scheme described in [**19**]; it is a special case of the "boundary-invariants" approach to tiling problems introduced in [**3**].

It will be conceptually helpful to extend a tiling T of the Aztec diamond to a tiling T^+ of the entire plane, by tiling the complement of the Aztec diamond by horizontal dominoes in the manner shown in Figure 3 for $n=3$. Let G be the graph with vertices $\{(a,b) \in \mathbf{Z}^2 : |a|+|b| \leq n+1\}$, and with an edge between (a,b) and (a',b') precisely when $|a-a'|+|b-b'|=1$. Color the lattice squares of $\mathbf{Z}^2$ in black-white checkerboard fashion, so that the line $\{(x,y) : x+y=n+1\}$ that bounds the upper right border of the Aztec diamond passes through only white squares. Call this the **standard** (or **even**) **coloring**. Orient each edge of G so that a black square lies to its left and a white square to its right; this gives the **standard orientation** of the graph G, with arrows circulating clockwise around white squares and counterclockwise around black squares. (Figure 4 shows the case $n=3$.) Write $u \to v$ if uv is an edge of G whose standard orientation is from u to v. Call $v=(a,b)$ a **boundary vertex** of G if $|a|+|b|=n$ or $n+1$, and let the **boundary cycle** be the closed zigzag path $(-n-1,0),(-n,0),(-n,1),(-n+1,1),(-n+1,2),...,(-1,n),(0,n),$ $(0,n+1),(0,n),(1,n),...,(n+1,0),...,(0,-n-1),...,(-n-1,0)$. Call the vertex $v=(a,b)$ **even** if it is the upper-left corner of a white square (i.e., if $a+b+n+1$ is even), and **odd** otherwise, so that in particular the four corner vertices $(-n-1,0),(n+1,0),(0,-n-1),(0,n+1)$ are even.

If one traverses the six edges that form the boundary of any domino, one will follow three edges in the positive sense and three edges in the negative sense. Also, every vertex v of G lies on the boundary of at least one domino in T^+. Hence if for definiteness one assigns "height" 0 to the leftmost vertex $(-n-1,0)$ of G, there is for each tiling T a unique way of assigning integer-valued heights $H_T(v)$ to all the vertices v of G, subject to the defining constraint that if the edge uv belongs to the boundary of some tile in T^+ with $u \to v$, then $H_T(v)=H_T(u)+1$. The resulting function $H_T(\cdot)$ is characterized by two properties:

(i) $H(v)$ takes on the successive values $0,1,2,...,2n+1,2n+2,2n+1,...,0,$ $...,2n+2,...,0$ as v travels along the boundary cycle of G;

(ii) if $u \to v$, then $H(v)$ is either $H(u)+1$ or $H(u)-3$.

The former is clear, since every edge of the boundary cycle is part of the boundary of a tile of T^+. To see that (ii) holds, note that if the edge uv belongs to T^+ (i.e. is part of the boundary of a tile of T^+), then $H(v)=H(u)+1$, whereas if uv does not belong to T^+ then it bisects a domino of T^+, in which case we see (by considering the other edges of that domino) that $H(v)=H(u)-3$.

In the other direction, notice that every height-function $H(\cdot)$ satisfying (i) and (ii) arises from a tiling T, and that the operation $T \mapsto H_T$ is reversible: given a function H satisfying (i) and (ii), we can place a domino covering every edge uv of G with $|H(u) - H(v)| = 3$, obtaining thereby a tiling of the Aztec diamond, which will coincide with the original tiling T in the event $H = H_T$. Thus there is a bijection between tilings of the Aztec diamond and height functions $H(\cdot)$ on the graph G that satisfy (i) and (ii). For a geometric interpretation of $H(\cdot)$, see [**19**].

Figure 5 shows the height-functions corresponding to two special tilings of the Aztec diamond, namely (a) the all-horizontal tiling $T_{\min}$ and (b) the all-vertical tiling $T_{\max}$. Since $H_T(v)$ is independent of T modulo 4, we are led to define the **reduced height**

$$h_T(v) = (H_T(v) - H_{T_{\min}}(v))/4;$$

parts (c) and (d) of Figure 5 show the reduced height-functions of $T_{\min}$ and $T_{\max}$, respectively. Lastly, we define the rank-statistic

$$r(T) = \sum_{v \in G} h_T(v).$$

It is easy to check that if one performs an elementary rotation on a 2-by-2 block centered at a vertex v (a "v-move" for short), the effect is to leave $h_T(v')$ alone for all $v' \neq v$ and to either increase or decrease $h_T(v)$ by 1; we call the move **raising** or **lowering** respectively.

We may now verify that $r(T)$ (as defined by the preceding equation) is equal to the number of elementary moves required to get from T to $T_{\min}$. Since $r(T_{\min}) = 0$, and since an elementary move merely changes the reduced height of a single vertex by ± 1, at least $r(T)$ moves are required to get from T to $T_{\min}$. It remains to check that for every tiling T there is a sequence of moves leading from T to $T_{\min}$ in which only $r(T)$ moves are made. To find such a sequence, let $T_0 = T$ and iterate the following operation for $i = 0, 1, 2, \ldots$: Select a vertex v_i at which $h_{T_i}(\cdot)$ achieves its maximum value. If $h_{T_i}(v_i) = 0$, then $T_i = T_{\min}$ and we are done. Otherwise, we have $h_{T_i}(v_i) > 0$, so that $H_{T_i}(v_i) > H_{T_{\min}}(v_i)$, with v_i not on the boundary of G (since h_{T_i} vanishes on the boundary). Reducing $H_{T_i}(v_i)$ by 4 preserves the legality of the height-labelling, and corresponds to performing a v_i-move on T_i, yielding a new tiling T_{i+1} with $r(T_{i+1}) = r(T_i) - 1$. By repeating this process, we continue to reduce the rank-statistic by 1 until the procedure terminates at $T_{r(T)} = T_{\min}$.

Thus we have shown that every tiling of the Aztec diamond may be reached from every other by means of moves of the sort described. This incidentally furnishes another proof that the number of dominoes of each orientation (horizontal or vertical) must be even, since this is clearly true of $T_{\min}$ and since every move annihilates two horizontal dominoes and creates two vertical ones, or vice versa.

The partial ordering on the set of tilings of an Aztec diamond given by height-functions has a pleasant interpretation in terms of a two-person game. Let T, T' be tilings of the Aztec diamond of order n. We give player A the tiling T and player B the tiling T'. On each round, A makes a rotation move, and B has the choice of either making the identical move (assuming it is available to her) or passing. Here, to make an "identical move" means to find an identically-situated 2-by-2 block in the identical orientation and give it a 90-degree twist. If, after a certain number of complete rounds (i.e. moves by A and counter-moves by B) A has solved her puzzle (that is, reduced the tiling to the all-horizontals tiling) while B has not, then A is deemed the winner; otherwise, B wins. Put $T' \preceq T$ if and only if B has a winning strategy in this game. It is easily checked (without even considering any facts about tilings) that the relation $\preceq$ is reflexive, asymmetric, and transitive. In fact, $T' \preceq T$ if and only if $h_T(v) \leq h_{T'}(v)$ for all $v \in G$. Moreover, the ideal strategy for either player is to make only lowering moves – though in the case $T' \preceq T$, it turns out that B can win by copying A whenever possible, regardless of whether such moves are lowering or raising.

3 Alternating sign matrices

An **alternating sign matrix** is a square matrix (n-by-n, say) all of whose entries are 1, -1, and 0, such that every row-sum and column-sum is 1, and such that the non-zero entries in each row and column alternate in sign; for instance

$$\begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & -1 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$

is a typical 4-by-4 alternating sign matrix. (For an overview of what is currently known about such matrices, see [**14**].) Let $\mathcal{A}_n$ denote the set of

n-by-n alternating sign matrices.

If A is an n-by-n alternating sign matrix with entries a_{ij} $(1 \leq i, j \leq n)$, we may define

$$a^*_{ij} = i \ + \ j \ - \ 2\left(\sum_{i'=1}^{i}\sum_{j'=1}^{j} a_{i'j'}\right)$$

for $0 \leq i, j \leq n$. We call the $(n+1)$-by-$(n+1)$ matrix A^* the **skewed summation** of A. (It is a variant of the "corner-sum matrix" of [**15**].) The matrices A^* that arise in this way are precisely those such that $a^*_{i0} = a^*_{0i} = i$ and $a^*_{in} = a^*_{ni} = n-i$ for $0 \leq i \leq n$, and such that adjacent entries of A' in any row or column differ by 1. Note that $a_{ij} = \frac{1}{2}(a^*_{i-1,j} + a^*_{i,j-1} - a^*_{i-1,j-1} - a^*_{i,j})$, so that an alternating sign matrix can be recovered from its skewed summation. Thus, the alternating sign matrix A defined above has

$$A^* = \begin{pmatrix} 0 & 1 & 2 & 3 & 4 \\ 1 & 2 & 1 & 2 & 3 \\ 2 & 1 & 2 & 1 & 2 \\ 3 & 2 & 3 & 2 & 1 \\ 4 & 3 & 2 & 1 & 0 \end{pmatrix}$$

as its skewed summation.

Our goal is to show that the domino tilings of the Aztec diamond of order n are in 1-to-1 correspondence with pairs (A, B) where $A \in \mathcal{A}_n$, $B \in \mathcal{A}_{n+1}$, and A, B jointly satisfy a certain "compatibility" relation. We will do this via the height-functions defined in the previous section.

Given a tiling T of the order-n Aztec diamond, we construct matrices A' and B' that record $H_T(v)$ for v odd and even, respectively (where $v = (x, y) \in G$ is even or odd according to the parity of $x + y + n + 1$). We let

$$a'_{ij} = H_T(-n + i + j, -i + j)$$

for $0 \leq i, j \leq n$ and

$$b'_{ij} = H_T(-n - 1 + i + j, -i + j)$$

for $0 \leq i,j \leq n+1$; thus, the tiling of Figure 6 gives the matrices

$$A' = \begin{pmatrix} 1 & 3 & 5 & 7 & 9 \\ 3 & 5 & 7 & 5 & 7 \\ 5 & 7 & 5 & 7 & 5 \\ 7 & 5 & 3 & 5 & 3 \\ 9 & 7 & 5 & 3 & 1 \end{pmatrix} \text{ and } B' = \begin{pmatrix} 0 & 2 & 4 & 6 & 8 & 10 \\ 2 & 4 & 6 & 4 & 6 & 8 \\ 4 & 6 & 8 & 6 & 4 & 6 \\ 6 & 4 & 6 & 4 & 6 & 4 \\ 8 & 6 & 4 & 2 & 4 & 2 \\ 10 & 8 & 6 & 4 & 2 & 0 \end{pmatrix}.$$

Note that the matrix-elements on the boundary of A' and B' are independent of the particular tiling T. Also note that in both matrices, consecutive elements in any row or column differ by exactly 2. Therefore, under suitable normalization, A' and B' can be seen as skewed summations of alternating sign matrices A and B. Specifically, by putting $a^*_{ij} = (a'_{ij} - 1)/2$ and $b^*_{ij} = b'_{ij}/2$, we arrive at matrices A^*, B^* which, under the inverse of the skewed summation operation, yield the matrices A, B that we desire:

$$A^* = \begin{pmatrix} 0 & 1 & 2 & 3 & 4 \\ 1 & 2 & 3 & 2 & 3 \\ 2 & 3 & 2 & 3 & 2 \\ 3 & 2 & 1 & 2 & 1 \\ 4 & 3 & 2 & 1 & 0 \end{pmatrix} \text{ and } B^* = \begin{pmatrix} 0 & 1 & 2 & 3 & 4 & 5 \\ 1 & 2 & 3 & 2 & 3 & 4 \\ 2 & 3 & 4 & 3 & 2 & 3 \\ 3 & 2 & 3 & 2 & 3 & 2 \\ 4 & 3 & 2 & 1 & 2 & 1 \\ 5 & 4 & 3 & 2 & 1 & 0 \end{pmatrix};$$

$$A = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & -1 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix} \text{ and } B = \begin{pmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & -1 & 1 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix}.$$

Conversely, A and B determine A' and B', which determine H_T, which determines T.

There is an easy way of reading off A and B from the domino-tiling T, without using height-functions. First, note that the even vertices in the interior of the Aztec diamond of order n are arranged in the form of a tilted n-by-n square. Also note that each such vertex is incident with 2, 3, or 4 dominoes belonging to the tiling T; if we mark each such site with a 1, 0, or -1 (respectively), we get the entries of A, where the upper-left corner of each matrix corresponds to positions near the left corner of the diamond. Similarly, the odd vertices of the Aztec diamond (including those on the boundary) form a tilted $(n+1)$-by-$(n+1)$ square. If we mark each such site with a -1, 0, or 1 according to whether it is incident with 2, 3, or 4 dominoes of the extended tiling T^+, we get the entries of B. (We omit the proof that this construction agrees with the one we gave earlier, since it is only the first one that we actually need.)

The legality constraint (ii) from the previous section tells us that for $1 \le i, j \le n$, the internal entries b'_{ij} of the matrix B' must be equal to

$$\begin{array}{llll} \text{either} & a'_{i-1,j-1} - 1 & \text{or} & a'_{i-1,j-1} + 3, \\ \text{either} & a'_{i-1,j} - 3 & \text{or} & a'_{i-1,j} + 1, \\ \text{either} & a'_{i,j-1} - 3 & \text{or} & a'_{i,j-1} + 1, \text{ and} \\ \text{either} & a'_{i,j} - 1 & \text{or} & a'_{i,j} + 3. \end{array}$$

Thus, in all but one of the six possible cases for the submatrix

$$\begin{pmatrix} a'_{i-1,j-1} & a'_{i-1,j} \\ a'_{i,j-1} & a'_{i,j} \end{pmatrix}$$

shown in Table 7, the value of b'_{ij} is uniquely determined; only in the case

$$\begin{pmatrix} 2k-1 & 2k+1 \\ 2k+1 & 2k-1 \end{pmatrix}$$

arising from $a_{ij} = 1$ does b'_{ij} have two possible values, namely $2k-2$ and $2k+2$.

It now follows that if we hold A fixed, the number of $(n+1)$-by-$(n+1)$ alternating sign matrices B such that the pair (A, B) yields a legal height function is equal to $2^{N_+(A)}$, where $N_+(A)$ is the number of $+1$'s in the n-by-n alternating sign matrix A. That is:

$$\mathrm{AD}(n) = \sum_{A \in \mathcal{A}_n} 2^{N_+(A)} \ . \tag{1}$$

Switching the roles of A and B, we may by a similar argument prove

$$\mathrm{AD}(n) = \sum_{B \in \mathcal{A}_{n+1}} 2^{N_-(B)} \ , \tag{2}$$

where $N_-(\cdot)$ gives the number of -1's in an alternating sign matrix. Replacing n by $n-1$ and B by A in (2), we get

$$\mathrm{AD}(n-1) = \sum_{A\in\mathcal{A}_n} 2^{N_-(A)} . \tag{3}$$

On the other hand, $N_+(A) = N_-(A) + n$ for all $A \in \mathcal{A}_n$, so (1) tells us that

$$\mathrm{AD}(n) = 2^n \sum_{A\in\mathcal{A}_n} 2^{N_-(A)} . \tag{4}$$

Combining (3) and (4), we derive the recurrence relation

$$\mathrm{AD}(n) = 2^n\,\mathrm{AD}(n-1),$$

which suffices to prove our formula for $\mathrm{AD}(n)$. (Mills, Robbins and Rumsey [**10**] prove

$$\sum_{A\in\mathcal{A}_n} 2^{N_-(A)} = 2^{n(n-1)/2}$$

as a corollary to their Theorem 2.)

In the remainder of this section, we discuss tilings and alternating sign matrices from the point of view of lattice theory. Specifically, we show that the tilings of an order-n Aztec diamond correspond to the lower ideals (or "down-sets") of a partially ordered set P_n, while the n-by-n alternating sign matrices correspond to the lower ideals of a partially ordered set Q_n, such that P_n consists of a copy of Q_n interleaved with a copy of Q_{n+1}. (For terminology associated with partially ordered sets, see [**17**].)

We start by observing that the set of legal height functions H on the order-n Aztec diamond is a poset in the obvious component-wise way, with $H_1 \geq H_2$ if $H_1(v) \geq H_2(v)$ for all $v \in G$. Moreover, the consistency conditions (i) and (ii) are such that if H_1 and H_2 are legal height-functions, then so are $H_1 \vee H_2$ and $H_1 \wedge H_2$, defined by $(H_1 \vee H_2)(v) = \max(H_1(v), H_2(v))$ and $(H_1 \wedge H_2)(v) = \min(H_1(v), H_2(v))$; thus our partially ordered set is actually a distributive lattice.

$\mathcal{A}_n$, the set of n-by-n alternating sign matrices, also has a lattice structure. Given $A_1, A_2 \in \mathcal{A}_n$, we form their skewed summations A_1^*, A_2^*, and declare $A_1 \geq A_2$ if every entry of A_1^* is greater than or equal to the corresponding entry of A_2^*. This partial ordering on alternating sign matrices is intimately connected with the partial ordering on tilings: if T_1, T_2 are tilings, then

$T_1 \geq T_2$ if and only if $A_1 \geq A_2$ and $B_1 \geq B_2$, where (A_i, B_i) is the pair of alternating sign matrices corresponding to the tiling T_i $(i = 1, 2)$.

For each vertex $v = (x, y)$ of the graph G associated with the order-n Aztec diamond (with $x, y \in \mathbf{Z}$, $|x| + |y| \leq n + 1$), let $m = H_{T_{\min}}(v)$ and $M = H_{T_{\max}}(v)$, so that $m, m+4, ..., M$ are the possible values of $H_T(v)$, and introduce points $(x, y, m), (x, y, m+4), ..., (x, y, M-4) \in \mathbf{Z}^3$ lying above the vertex $v = (x, y)$. (Note that if (x, y) is on the boundary of G, $m = M$, so the set of points above (x, y) is empty.) Let P denote the set of all such points as v ranges over the vertex-set of G. We make P a directed graph by putting an edge from $(x, y, z) \in P$ to $(x', y', z') \in P$ provided $z = z' + 1$ and $|x - x'| + |y - y'| = 1$; we then make P a partially ordered set by putting $(x, y, z) \geq (x', y', z')$ if there is a sequence of arrows leading from (x, y, z) to (x', y', z').

To each height-function H we may assign a subset $I_H \subseteq P$, with $I_H = \{(x, y, z) \in P : z < H(x, y)\}$. This operation is easily seen to be a bijection between the legal height-functions H and the lower ideals of the partially ordered set P. Indeed, the natural lattice structure on the set of height-functions H (with $H_1 \leq H_2$ precisely if $H_1(v) \leq H_2(v)$ for all $v \in G$) makes it isomorphic to the lattice $J(P)$ of lower ideals of P, and the rank $r(T)$ of a tiling T (as defined above) equals the rank of H_T in the lattice, which in turn equals the cardinality of I_{H_T}.

Note that for all $(x, y, z) \in P$, $x + y + z \equiv n + 1 \bmod 2$. The poset P decomposes naturally into two complementary subsets P^{even} and P^{odd}, where a point $(x, y, z) \in P$ belongs to P^{even} if z is even and P^{odd} if z is odd. The vertices of P^{even} form a regular tetrahedral array of side n, resting on a side (as opposed to a face); that is, it consists of a 1-by-n array of nodes, above which lies a 2-by-$(n-1)$ array of nodes, above which lies a 3-by-$(n-2)$ array of nodes, and so on, up to the n-by-1 array of nodes at the top. The partial ordering of P restricted to P^{even} makes P^{even} a poset in its own right, with (x, y, z) covering (x', y', z') when $z = z' + 2$ and $|x - x'| = |y - y'| = 1$. Similarly, the vertices of P^{odd} form a tetrahedral array of side $n - 1$; each vertex of P^{odd} lies at the center of a small tetrahedron with vertices in P^{even}. P^{odd}, like P^{even}, is a poset in itself, with (x, y, z) covering (x', y', z') when $z = z' + 2$ and $|x - x'| = |y - y'| = 1$.

Our correspondence between height-functions H_T and pairs (A, B) of alternating sign matrices tells us that $\mathcal{A}_n$, as a lattice, is isomorphic to $J(P_n^{\text{odd}})$, while $\mathcal{A}_{n+1}$ is isomorphic to $J(P_n^{\text{even}})$. Indeed, under this isomorphism, $A \in \mathcal{A}_n$ and $B \in \mathcal{A}_{n+1}$ are compatible if and only if the union of the down-sets of P^{even} and P^{odd} corresponding to A and B is a down-set of $P = P^{\text{even}} \cup P^{\text{odd}}$. (This coincides with the notion of compatibility given in [**15**].) If we let Q_n denote the tetrahedral poset P_n^{odd} (so that P_n^{even} is isomorphic to Q_{n+1}), then we see that P_n indeed consists of a copy of Q_n interleaved with a copy of Q_{n+1}.

As an aid to visualizing the poset P and its lower ideals, we may use stacks of marked 2-by-2-by-2 cubes resting on a special multi-level tray. The bottom face of each cube is marked by a line joining midpoints of two opposite edges, and the top face is marked by another such line, skew to the mark on the bottom face. (See Figure 8.) These marks constrain the ways in which we allow ourselves to stack the cubes. To enforce these constraints, whittle away the edges of the cube on the top and bottom faces that are parallel to the marks on those faces, and replace each mark by a protrusion, as in Figure 9; the rule is that a protrusion on the bottom face of a cube must fit into the space between two whittled-down edges (or between a whittled-down edge and empty space). The only exception to this rule is at the bottom of the stack, where the protrusions must fit into special furrows in the tray. Figure 10 shows the tray in the case $n = 4$; it consists of four levels, three of which float in mid-air. On the bottom level, the outermost two of the three gently sloping parallel lines running from left to right should be taken as protrusions, and the one in between should be taken as a furrow. Similarly, in the higher levels of the tray, the outermost lines are protrusions and the innermost two are furrows. We require that the cubes that rest on the table must occupy only the n obvious discrete positions; no intermediate positions are permitted. Also, a cube cannot be placed unless its base is fully supported, by the tray, a tray and a cube, or two cubes.

In stacking the cubes, one quickly sees that in a certain sense one has little freedom in how to proceed; any stack one can build will be a subset of the stack shown in Figure 11 in the case $n = 4$. Indeed, if one partially orders the cubes in Figure 11 by the transitive closure of the relation "is resting on", then the poset that results is the poset P defined earlier, and the admissible stacks correspond to lower ideals of P in the obvious way. Moreover, the markings visible to an observer looking down on the stack yield a picture of the domino tiling that corresponds to that stack.

4 Monotone triangles

Let A^* be the skewed summation of an n-by-n alternating sign matrix A. Notice that the ith row ($0 \leq i \leq n$) begins with an i and ends with an $n-i$, so that reading from left to right, we must see i descents and $n-i$ ascents; that is, there are exactly i values of j in $\{1, 2, ..., n\}$ satisfying $a'_{i,j} = a'_{i,j-1} - 1$, and the remaining $n-i$ values of j satisfy $a'_{i,j} = a'_{i,j-1} + 1$. Form a triangular array whose ith row ($1 \leq i \leq n$) consists of those values of j for which $a'_{i,j} = a'_{i,j-1} - 1$. E.g., for

$$A = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & -1 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \end{pmatrix} \text{ and } A' = \begin{pmatrix} 0 & 1 & 2 & 3 & 4 \\ 1 & 2 & 1 & 2 & 3 \\ 2 & 1 & 2 & 1 & 2 \\ 3 & 2 & 3 & 2 & 1 \\ 4 & 3 & 2 & 1 & 0 \end{pmatrix}$$

we get the triangle

$$\begin{array}{ccccccc} & & & 2 & & & \\ & & 1 & & 3 & & \\ & 1 & & 3 & & 4 & \\ 1 & & 2 & & 3 & & 4 \end{array}$$

Note that j occurs in the ith row of the monotone triangle exactly if the sum of the first i entries in column j of the alternating sign matrix is 1.

A **monotone triangle** of size n is a triangular array of natural numbers with strict increase from left to right along its n rows and with non-strict increase from left to right along its diagonals, as in the array above. If the bottom row of a monotone triangle is $1\ 2\ ...\ n$, we call the array a **complete monotone triangle**. It is not hard to show that the preceding construction gives a bijection between the n-by-n alternating sign matrices and the complete monotone triangles of size n. Moreover, the $+1$'s in the alternating sign matrix correspond to entries in some row of the triangle that do not occur in the preceding row.

It follows from the foregoing that AD(n) is the sum, over all complete monotone triangles of size n, of 2 to the power of the number of entries in the monotone triangle that do not occur in the preceding row. Since a monotone triangle of size n has exactly $n(n+1)/2$ entries, we may divide both sides of the equation $\text{AD}(n) = 2^{n(n+1)/2}$ by $2^{n(n+1)/2}$ and paraphrase it as the claim that the sum, over all complete monotone triangles of size n, of $\frac{1}{2}$ to the power of the number of entries in the monotone triangle that *do* occur in the preceding row, is precisely 1.

Define the weight of a monotone triangle (of any size) as $\frac{1}{2}$ to the power of the number of entries that appear in the preceding row, and let $W(a_1, a_2, ..., a_k)$ be the sum of the weights of the monotone triangles of size k with bottom row $a_1\ a_2\ ...\ a_k$. (For now we may assume $a_1 < a_2 < ... < a_k$, although we will relax this restriction shortly.) Our goal is to prove that $W(1, 2, ..., n) = 1$ for all n.

To this end, observe that we have the recurrence relation

$$W(a_1, a_2, ..., a_n) = \sum_{b_1=a_1}^{a_2}{}^{*} \sum_{b_2=a_2}^{a_3}{}^{*} ... \sum_{b_{n-1}=a_{n-1}}^{a_n}{}^{*} W(b_1, b_2, ..., b_{n-1}) \qquad (5)$$

for all n, where $\sum^*$ is the modified summation operator

$$\sum_{i=r}^{s}{}^{*} f(i) = \frac{1}{2}f(r) + f(r+1) + f(r+2) + ... + f(s-1) + \frac{1}{2}f(s)\ ;$$

for the number of factors of $\frac{1}{2}$ that contribute to the coefficient of $W(b_1, b_2, ..., b_{n-1})$ is exactly the number of b_i's that also occur among the a_i's. Observe that the operator $\sum^*$ resembles definite integration in that

$$\sum_{i=r}^{s}{}^{*} f + \sum_{i=s}^{t}{}^{*} f = \sum_{i=r}^{t}{}^{*} f \qquad (6)$$

for $r < s < t$. Indeed, if we extend $\sum^*$ by defining

$$\sum_{i=r}^{r}{}^{*} f = 0$$

for all r and

$$\sum_{i=r}^{s}{}^{*} f = -\sum_{i=s}^{r}{}^{*} f$$

for $r > s$, then (6) holds for all integers r, s, t. Hence, starting from the base-relation $W(a_1) = 1$, equation (5) can be applied iteratively to define $W(\cdot)$ as a function of $a_1, a_2, ..., a_n$, regardless of whether $a_1 < a_2 < ... < a_n$ or not.

Notice that if $f(x) = x^m$, then

$$\sum_{r=s}^{t}{}^{*} f(r)$$

is a polynomial in s and t of degree $m+1$, of the form

$$\frac{t^{m+1} - s^{m+1}}{m+1} + \text{ terms of lower order.}$$

More generally, if $f(x, y, z, ...)$ is a polynomial in $x, y, z, ...$ with a highest-order term $cx^m y^{m'} z^{m''}...$, then

$$\sum_{r=s}^{t}{}^{*} f(r, y, z, ...)$$

is a polynomial in $s, t, y, z, ...$ of degree $\deg f + 1$, with highest-order terms

$$\frac{c}{m+1} t^{m+1} y^{m'} z^{m''} \ldots \quad \text{and} \quad -\frac{c}{m+1} s^{m+1} y^{m'} z^{m''} \ldots \; .$$

We will now use (5) and (6) to prove the general formula

$$W(a_1, a_2, ..., a_n) = \prod_{1 \le i < j \le n} \frac{a_j - a_i}{j - i} \; . \tag{7}$$

(This immediately yields $W(1, 2, ..., n) = 1$, which as we have seen implies $\mathrm{AD}(n) = 2^{n(n+1)/2}$.) Formula (7) is equivalent to Theorem 2 in [**11**], but we offer our own proof.

The proof is by induction. When $n = 1$, we have $W(a_1) = 1$, so that (7) is satisfied. Suppose now that we have

$$W(b_1, b_2, ..., b_{n-1}) = \prod_{1 \le i < j \le n-1} \frac{b_j - b_i}{j - i}$$

for all $b_1, b_2, ..., b_{n-1}$. Since $W(b_1, b_2, ..., b_{n-1})$ is a polynomial of degree $(n-1)(n-2)/2$ with a highest-order term

$$\frac{b_2^1\, b_3^2 \,\cdots\, b_{n-1}^{n-2}}{1!\,2! \,\cdots\, (n-2)!} \ ,$$

the recurrence relation (5) and the observations made in the preceding paragraph imply that $W(a_1, a_2, ..., a_n)$ is a polynomial of degree

$$(n-1)(n-2)/2 + (n-1) = n(n-1)/2$$

with a highest-order term

$$\frac{a_2^1\, a_3^2 \,\cdots\, a_n^{n-1}}{1!\,2! \,\cdots\, n!} \ .$$

To complete the proof, we need only show that $W(a_1, a_2, ..., a_n)$ is skew-symmetric in its arguments; for this implies that it is divisible by $(a_2 - a_1)(a_3 - a_1) \cdots (a_n - a_{n-1})$, a polynomial of the same degree (namely $n(n-1)/2$) as itself, and a comparison of the coefficients of leading terms yields (7).

It suffices to show that interchanging any two consecutive arguments of W changes the sign of the result. For convenience, we illustrate with $n = 4$:

$$\begin{aligned} W(a_2, a_1, a_3, a_4) &= \sum_{b_1=a_2}^{a_1}{}^{*} \sum_{b_2=a_1}^{a_3}{}^{*} \sum_{b_3=a_3}^{a_4}{}^{*} W(b_1, b_2, b_3) \\ &= \left(- \sum_{b_1=a_1}^{a_2}{}^{*} \right) \left(\sum_{b_2=a_1}^{a_2}{}^{*} + \sum_{b_2=a_2}^{a_3}{}^{*} \right) \left(\sum_{b_3=a_3}^{a_4}{}^{*} \right) W(b_1, b_2, b_3) \ . \end{aligned}$$

The skew-symmetry of $W(b_1, b_2, b_3)$ kills off one of the two terms:

$$\begin{aligned} & \sum_{b_1=a_1}^{a_2}{}^{*} \sum_{b_2=a_1}^{a_2}{}^{*} \sum_{b_3=a_3}^{a_4}{}^{*} W(b_1, b_2, b_3) \\ = \ & \sum_{b_2=a_1}^{a_2}{}^{*} \sum_{b_1=a_1}^{a_2}{}^{*} \sum_{b_3=a_3}^{a_4}{}^{*} W(b_2, b_1, b_3) \\ & \text{(by re-labelling)} \\ = \ & \sum_{b_1=a_1}^{a_2}{}^{*} \sum_{b_2=a_1}^{a_2}{}^{*} \sum_{b_3=a_3}^{a_4}{}^{*} W(b_2, b_1, b_3) \end{aligned}$$

$$
\begin{aligned}
& \quad \text{(by commutativity)} \\
= & \quad -\sum_{b_1=a_1}^{a_2}{}^{*} \sum_{b_2=a_1}^{a_2}{}^{*} \sum_{b_3=a_3}^{a_4}{}^{*} W(b_1,b_2,b_3) \\
& \quad \text{(by skew-symmetry)},
\end{aligned}
$$

implying that the term vanishes. Hence

$$
\begin{aligned}
W(a_2,a_1,a_3,a_4) &= -\sum_{b_1=a_1}^{a_2}{}^{*} \sum_{b_2=a_2}^{a_3}{}^{*} \sum_{b_3=a_3}^{a_4}{}^{*} W(b_1,b_2,b_3) \\
&= -W(a_1,a_2,a_3,a_4)\ ,
\end{aligned}
$$

as claimed. Similarly, $W(a_1,a_2,a_4,a_3) = -W(a_1,a_2,a_3,a_4)$. A slightly more complicated calculation, involving a sum of four terms of which three vanish, gives $W(a_1,a_3,a_2,a_4) = -W(a_1,a_2,a_3,a_4)$. The argument for the skew-symmetry of $W(a_1,a_2,...,a_n)$ is much the same for n in general, although the notation is more complex; we omit the details.

Having shown that W is skew-symmetric in its arguments, we have completed the proof of (7), which yields the formula for $\mathrm{AD}(n)$ as a consequence.

Some remarks are in order. First, it is noteworthy that

$$
\prod_{1\le i<j\le n} \frac{a_j-a_i}{j-i}
$$

is an integer provided $a_1,...,a_n$ are; this can be proved in a messy but straightforward manner by showing that every prime p must divide the numerator at least as many times as it divides the denominator. Alternatively, one can show that this product is equal to the determinant of the n-by-n matrix whose i,jth entry is the integer

$$
\binom{a_i}{j-1}
$$

(see [**13**] and [**16**]).

Second, formula (7) has a continuous analogue: If we take $V(x)=1$ for all real x and inductively define

$$
V(x_1,x_2,...,x_n) = \int_{x_1}^{x_2}\int_{x_2}^{x_3}\cdots\int_{x_{n-1}}^{x_n} V(y_1,y_2,...,y_{n-1})\,dy_{n-1}\,\cdots\,dy_2\,dy_1\ ,
$$

then essentially the same argument shows that

$$V(x_1, x_2, ..., x_n) = \prod_{1 \le i < j \le n} \frac{x_j - x_i}{j - i} .$$

This has the following probabilistic interpretation: Given n real numbers $x_1 < x_2 < ... < x_n$, let $X_{i,i} = x_i$ for $1 \le i \le n$, and for all $1 \le i < j \le n$ let $X_{i,j}$ be a number chosen uniformly at random in the interval $[x_i, x_j]$. Then the probability that $X_{i,j} \le X_{i+1,j}$ and $X_{i,j} \le X_{i,j+1}$ for all suitable i, j is

$$\prod_{1 \le i < j \le n} \frac{1}{j - i} = \frac{1}{1!\,2! \cdots (n-1)!} .$$

We do not know a more direct proof of this fact than the one outlined here.

Third, the usual (unstarred) summation operator does not satisfy a relation like (6), so the method used here will not suffice to count unweighted monotone triangles. (Mills, Robbins, and Rumsey offer abundant evidence that the number of complete monotone triangles of size n is

$$\prod_{k=0}^{n-1} \frac{(3k+1)!}{(n+k)!} ,$$

but no proof has yet been found.) However, the operators

$$\sum_{i=r}^{s}{}^{L} = \sum_{i=r}^{s-1}$$

and

$$\sum_{i=r}^{s}{}^{R} = \sum_{i=r+1}^{s}$$

do satisfy an analogue of (6), and one can exploit this to give streamlined proofs of some formulas in the theory of plane partitions; details will appear elsewhere.

Fourth, we should note that the function $W(a_1, a_2, ..., a_m)$ has significance for tilings of the Aztec diamond of order n, even outside the case with $m = n$ and $a_i = i$ for $1 \le i \le m$. Suppose $m \le n$ and $a_m \le n$, and let Π be the path in the graph G that starts at $(-m, n-m)$ whose $2j-1$st and $2j$th steps head south and east respectively if $j \in \{a_1, ..., a_n\}$ and otherwise head east and south respectively, for $1 \le j \le n$, ending at the vertex $(n-m,-m)$; Figure 12(a) shows Π when $n=4$, $m=2$, $(a_1,a_2)=(2,3)$. It is not hard to show that the number of domino tilings of the portion of the Aztec diamond that lies above Π is $2^{m(m+1)/2}W(a_1,a_2,...,a_m)$.

Fifth (and last), we should note that the role played by the matrix A at the beginning of the section (in expressing $\mathrm{AD}(n)$ in terms of a weighted sum over complete monotone triangles of size n) could have been played just as well by the matrix B, giving rise to an alternative formula expressing $\mathrm{AD}(n)$ as the sum, over all complete monotone triangles of size $n+1$, of 2 to the power of the number of entries above the bottom row that do not occur in the succeeding row. But, dividing by $2^{n(n+1)/2}$, we reduce the claim $\mathrm{AD}(n) = 2^{n(n+1)/2}$ to the same claim as before (the sum of the weights of all the fractionally weighted complete monotone triangles of any given size is equal to 1). This gives a second significance of $W(\cdots)$ for tilings of the Aztec diamond. Specifically, suppose $m \le n+1$ and $a_m \le n$, and let Π be the path in the graph G that starts at $(-m,n+1-m)$ whose $2j-1$st and $2j$th steps head east and south respectively if $j \in \{a_1,...,a_n\}$ and otherwise head south and east respectively, for $1 \le j \le n+1$, ending at the vertex $(n+1-m,-m)$; Figure 12(b) shows Π when $n=4$, $m=2$, $(a_1,a_2)=(2,3)$. It can be shown that the number of domino tilings of the portion of the Aztec diamond that lies above Π is $2^{m(m-1)/2}W(a_1,a_2,...,a_m)$.

5 Grassmann algebras

The resemblance between the formula for W and the Weyl dimension formula for representations of $GL(n)$ is not coincidental. In fact, the identity

$$\sum_A 2^{N(A)} = 2^{n(n-1)/2}$$

can be proved by pure representation theory. The idea is to relate the rules for consecutive rows in Gelfand triangles to the decomposition of $GL(n)$-representations as $GL(n-1) \times GL(1)$ representations.

Let V be a finite-dimensional vector space, $\Lambda^i(V)$ the ith exterior power of V, and

$$\mathrm{Grass}(V) = \bigoplus_{i=0}^{n} \Lambda^i(V),$$

the Grassmann algebra generated by V. It is elementary that if V and W are finite-dimensional vector spaces,

$$\Lambda^2(V \oplus W) = \Lambda^2(V) \oplus \Lambda^2(W) \oplus (V \otimes W),$$

$$\mathrm{Grass}(V \oplus W) = \mathrm{Grass}(V) \otimes \mathrm{Grass}(W).$$

Writing $\mathrm{G}(V)$ for $\mathrm{Grass}(\Lambda^2(V))$, we get $\dim(\mathrm{G}(V)) = 2^{\binom{\dim(V)}{2}}$ and

$$\mathrm{G}(V \oplus W) = \mathrm{G}(V) \otimes \mathrm{G}(W) \otimes \mathrm{Grass}(V \otimes W).$$

We now recall the Cartan-Weyl theory of weights of irreducible representations of Lie groups, in the case of $GL(n)$ (due to Schur); for more details, see [**5**]. If $V = \mathbf{C}^n$, then $GL(V) = GL(n, \mathbf{C})$ contains the group T of diagonal matrices $diag(x_1, ..., x_n)$. The analytic homomorphisms $T \to \mathbf{C}^*$ are precisely the Laurent monomials $x_1^{a_1} \cdots x_n^{a_n}$, $a_i \in \mathbf{Z}$. If ρ is a finite-dimensional (analytic) representation of $GL(n, \mathbf{C})$, its restriction to T is a direct sum of 1-dimensional analytic representations (called *weights*), and the restriction of the trace of ρ to T is a Laurent polynomial in the x_i; we represent a weight of ρ by the sequence of exponents occurring in the corresponding Laurent monomial in $tr(\rho|_T)$. For instance, the trace function of the identity representation is the sum of the diagonal elements, $x_1 + \cdots + x_n$, so the weights are the basis vectors $(0, ..., 0, 1, 0, ..., 0)$. The operations of linear algebra can be translated into operations on trace polynomials. Thus, the trace of a direct sum of representations is the sum of the traces, the trace of a tensor product is the product of traces, the trace of the kth exterior power is the kth elementary symmetric function of the constituent monomials, and so on. The irreducible representations ρ of $GL(n, \mathbf{C})$ are indexed by *dominant weights* $\lambda = (\lambda_1, ..., \lambda_n)$, where $\lambda_i \in \mathbf{Z}$ and $\lambda_1 \leq ... \leq \lambda_n$; among all weights occurring in $\rho|_T$ satisfying this inequality, λ has the greatest norm. For instance, the dominant weight of the identity representation is $(0, 0, ..., 0, 1)$.

We set $a_i = \lambda_i + i$, so the finite-dimensional irreducible representations of $GL(n, \mathbf{C})$ are indexed by $(a_1, ..., a_n)$ where $a_i \in \mathbf{Z}$ and $a_1 < ... < a_n$. The Weyl character formula for $GL(n)$ says that the trace function for $\rho(a_1, ..., a_n)$ is

$$\frac{\sum_{\sigma \in S_n} \mathrm{sgn}(\sigma)\, {x_1}^{a_{\sigma(1)}} \cdots {x_n}^{a_{\sigma(n)}}}{\prod_{1 \leq i \leq n} x_i \ \prod_{1 \leq j < i \leq n} (x_i - x_j)}.$$

The numerator of this expression can be written

$$\det \begin{pmatrix} x_1^{a_1} & x_2^{a_1} & \dots & x_n^{a_1} \\ x_1^{a_2} & x_2^{a_2} & \dots & x_n^{a_2} \\ \vdots & \vdots & \ddots & \vdots \\ x_1^{a_n} & x_2^{a_n} & \dots & x_n^{a_n} \end{pmatrix} .$$

Subtracting $x_1^{a_{i+1}-a_i}$ times row i from row $i+1$, for $i = n-1, n-2, ..., 1$, we obtain

$$x_1^{a_1} \sum_{\tau \in S_{n-1}} \operatorname{sgn}(\tau) \prod_{i=1}^{n-1} \left(x_{i+1}^{a_{\tau(i)+1}} - x_{i+1}^{a_{\tau(i)}} x_1^{a_{\tau(i)+1}-a_{\tau(i)}} \right) .$$

The trace functions for the representations V, $\Lambda^2(V)$, and $\mathrm{G}(V)$ (that is, for the action of $GL(n)$ on these spaces induced by the action of $GL(n)$ on V) are given by $\sum_{1\le i\le n} x_i$, $\sum_{1\le j<i\le n} x_i x_j$, and $\sum_{1\le j<i\le n}(1+x_i x_j)$, respectively. The trace function for $\rho(a_1,...,a_n)\otimes \mathrm{G}(V)$ is therefore

$$x_1^{a_1} \sum_{\tau \in S_{n-1}} \operatorname{sgn}(\tau) \prod_{i=1}^{n-1} \left(x_{i+1}^{a_{\tau(i)+1}} - x_{i+1}^{a_{\tau(i)}} x_1^{a_{\tau(i)+1}-a_{\tau(i)}} \right) \prod_{i=1}^{n} \frac{1}{x_i} \prod_{1\le j<i\le n} \frac{1+x_i x_j}{x_i - x_j};$$

this is equal to

$$x_1^{a_1-1} \sum_{\tau \in S_{n-1}} \operatorname{sgn}(\tau) \prod_{i=1}^{n-1} (\Sigma_1(i,\tau) + \Sigma_2(i,\tau)) \prod_{i=2}^{n} \frac{1}{x_i} \prod_{2\le j<i\le n} \frac{1+x_i x_j}{x_i - x_j},$$

where

$$\begin{aligned} \Sigma_1(i,\tau) &= \sum_{k=a_{\tau(i)}}^{a_{\tau(i)+1}-1} x_{i+1}^{k} x_1^{a_{\tau(i)+1}-1-k} , \\ \Sigma_2(i,\tau) &= \sum_{\ell=a_{\tau(i)}+1}^{a_{\tau(i)+1}} x_{i+1}^{\ell} x_1^{a_{\tau(i)+1}-1-\ell} . \end{aligned}$$

Viewing $GL(n-1,\mathbf{C})$ as the subgroup of $GL(n,\mathbf{C})$ consisting of all matrices of the form $\begin{pmatrix} 1 & 0 \\ 0 & M \end{pmatrix}$, we can restrict $\rho(a_1,...,a_n)\otimes \mathrm{G}(\mathbf{C}^n)$ to $GL(n-1)$. At the level of traces on the diagonal, this amounts to setting $x_1 = 1$, to obtain

$$\left(2 \sum_{b_1=a_1}^{a_2}{}^{*}\right) \cdots \left(2 \sum_{b_{n-1}=a_{n-1}}^{a_n}{}^{*}\right) \sum_{\tau \in S_{n-1}} \operatorname{sgn}(\tau) x_2^{b_{\tau(1)}} \cdots x_n^{b_{\tau(n-1)}} \prod_{i=2}^{n} \frac{1}{x_i} \prod_{2\le j<i\le n} \frac{1+x_i x_j}{x_i - x_n}.$$

(As in the preceding section, the notation $\sum^*$ indicates a sum where the endpoints are counted with multiplicity $\frac{1}{2}$.) This is visibly the sum of the traces of the $GL(n-1)$-representations

$$\rho(b_1, ..., b_{n-1}) \otimes \mathrm{G}(\mathbf{C}^{n-1}),$$

counted with appropriate multiplicities. In fact, since two representations of $GL(n-1)$ are the same if and only if their trace polynomials coincide, this gives a formula for the restriction of the representation $\rho(a_1, ..., a_n) \otimes \mathrm{G}(\mathbf{C}^n)$ to $GL(n-1)$. Iterating this process, we see that $W(1, 2, ..., n)$ (as defined in the previous section) is the value obtained by substituting $x_1 = x_2 = ... = x_n = 1$ in the trace function of $\mathrm{G}(\mathbf{C}^n)$ viewed as a $GL(n)$-representation, or in other words, the trace function of $\mathrm{G}(\mathbf{C}^n)$ on $GL(0) = 1$, which is simply the dimension of $\mathrm{G}(\mathbf{C}^n)$.

6 Domino shuffling

The even (or standard) coloring of the Aztec diamond, as defined earlier, is the black-white checkerboard coloring in which the interior squares along the northeast border are black. In this section, it will be convenient to also consider the other checkerboard coloring, which we call **odd**. We will continue to call a vertex of a checkerboard-colored region **even** if it is the upper-left corner of a white square and **odd** otherwise, only now this notion depends on the checkerboard coloring chosen as well as on the coordinates of the vertex.

In general, a union of squares in a bi-colored checkerboard will be called *even* if the leftmost square in its top row is white, and *odd* if that square is black. Thus, the left half of Figure 13shows an even Aztec diamond, an even 2-by-2 block, and two even dominoes (along with an even vertex), while the right half of Figure 13shows odd objects of the same kind.

Given a tiling of a colored region by dominoes, we may remove all the odd blocks to obtain an **odd-deficient tiling.** In general, an odd-deficient domino tiling of a region in the plane is a partial tiling that has no odd blocks and that can be extended to a complete tiling of that region by adding only odd blocks. An odd-deficient tiling of the Aztec diamond of order n with its even coloring is uniquely determined by the heights of its even vertices, as recorded in the matrix B of section 3; thus, these odd-deficient tilings are in one-one correspondence with alternating sign matrices of order $n+1$.

Given a partial tiling $\tilde{T}$ of the plane, let $U_{\tilde{T}}$ be the union of the dominoes belonging to $\tilde{T}$. Observe that if $\tilde{T}$ is odd-deficient, then the boundary of $U_{\tilde{T}}$ has corners only at odd vertices.

The functions $v(T)$ and $r(T)$ defined earlier can be expressed in the form

$$v(T) = \sum_{d \in T} v(d)$$

and

$$r(T) = \sum_{d \in T} r(d)$$

for suitable functions $v(\cdot)$ and $r(\cdot)$ on the set of dominoes, which we now define. If the domino d is horizontal, let $v(d) = r(d) = 0$; if d is vertical, let $v(d) = \frac{1}{2}$ and let $r(d)$ be assigned according to the location of the center of d following the pattern set down in Figure 14 for the case $n = 3$. (More formally, we may declare that if d is the vertical domino with upper-left corner at (i, j), then $r(d) = (-1)^{i+j+n}(i + n + 1)$.) Clearly $v(T)$ is the sum of $v(d)$ over all dominoes $d \in T$. As for $r(T)$, note that

$$r(T_{\min}) = 0 = \sum_{d \in T_{\min}} r(d) \ ;$$

also note that a move that increases $h(T)$ by 1 either creates two vertical dominoes d_1, d_2 satisfying $r(d_1) + r(d_2) = 1$ or annihilates two vertical dominoes d_1, d_2 satisfying $r(d_1)+r(d_2) = -1$. Thus by induction $r(T) = \sum_{d \in T} r(d)$ for all tilings T.

We therefore have

$$\mathrm{AD}(n; x, q) = \sum_{T} \prod_{d \in T} x^{v(d)} q^{r(d)}.$$

We now prove

$$\mathrm{AD}(n; x, q) = \prod_{k=0}^{n-1} (1 + xq^{2k+1})^{n-k}$$

using a process called **domino-shuffling**, which is a certain involution on the set of odd-deficient tilings of an infinite checkerboard. If d is domino on a colored region, we define $S(d)$, the **shuffle** of d, as the domino obtained by moving d one unit to the left or up if it is even and one unit to the right or down if it is odd. (See Figure 15.) Graphically, one can put an arrow joining the two non-corner vertices on the boundary of d, pointing from the even vertex to the odd vertex; this indicates the direction in which d will shuffle.

Clearly S is an involution on the set of dominoes on an infinite checkerboard. Two dominoes form an odd block if and only if each is the shuffle of the other; if d and $S(d)$ are horizontal, then $r(d) + r(S(d)) = 0$, while if d and $S(d)$ are vertical, then $r(d) + r(S(d)) = -1$.

Given a partial tiling $\tilde{T}$ we define $S(\tilde{T})$, the **shuffle** of $\tilde{T}$, to be the collection of all $S(d)$ with $d \in \tilde{T}$.

LEMMA: Domino shuffling is an involution on the odd-deficient tilings of an infinite checkerboard.

Proof: Let $\tilde{T}$ be an odd-deficient tiling of the plane, with T an extension to a true tiling of the plane. We first show that $S(\tilde{T})$ is a partial tiling, that is, that no two dominoes of $S(\tilde{T})$ overlap. Assume otherwise, and suppose that a white square s is covered by two dominoes in $S(\tilde{T})$. That is, $S(\tilde{T})$ contains two of the four dominoes a, b, c, d shown in Figure 16 (with arrows indicating the directions in which they shuffle). There are six cases to be considered and ruled out.

$a, b \in S(\tilde{T})$: $\tilde{T}$ must contain the dominoes $S^{-1}(a) = S(a)$ and $S^{-1}(b) = S(b)$. But $S(a)$ and $S(b)$ overlap (see Figure 17(a)).

$c, d \in S(\tilde{T})$: Same reasoning.

$a, c \in S(\tilde{T})$: $S(a)$, $S(c) \in \tilde{T}$ (see Figure 17(b)). The full tiling T must cover s but cannot include b or d (since T already includes $S(a)$ and $S(c)$ which conflict with those two dominoes); hence T must include a or c. But in the former case, $a \in T$ forms an odd block with $S(a) \in T$, so that $S(a) \notin \tilde{T}$ after all; and the case $c \in T$ leads to a similar contradiction.

$b, d \in S(\tilde{T})$: Same reasoning.

$a, d \in S(\tilde{T})$: Same reasoning as in the preceding two cases, though the geometry is somewhat different (see Figure 17(c)).

$b, c \in S(\tilde{T})$: Same reasoning.

Hence a white square cannot be covered by two dominoes of $S(\tilde{T})$. The proof for black squares is similar. Therefore, $S(\tilde{T})$ is a partial tiling of the checkerboard.

We must also show that $S(\tilde{T})$ is odd-deficient. $S(\tilde{T})$ cannot contain any odd blocks, because the inverse shuffle (which is the same as the shuffle) of an odd block is an odd block. It remains to show that the boundary of $U_{S(\tilde{T})}$ has corners only at odd vertices. Let v be an even vertex. It is easily checked that v is a corner of $U_{\tilde{T}}$ if and only if $U_{\tilde{T}}$ contains unequal numbers of black squares and white squares adjacent to v (and similarly for $U_{S(\tilde{T})}$). A domino $d \in \tilde{T}$ may cover, of the four squares adjacent to v, one black square, one white square, or one square of each color. In these three cases, $S(d)$ covers one white square, one black square, or no squares at all, respectively. Thus the even vertex v could be a corner of $U_{S(\tilde{T})}$ only if it was already a corner of $U_{\tilde{T}}$. But we assumed $\tilde{T}$ was odd-deficient, so that its only corners were at odd vertices. □

Assume now that $\tilde{T}$ is an odd-deficient tiling, not of the entire plane, but of the order-$(n-1)$ Aztec diamond. We can use the above to show that $S(\tilde{T})$ is an odd-deficient tiling of the order-n diamond. It is clear that for every domino $d \in \tilde{T}$, $S(d)$ lies in the order-n diamond; what is less pictorially obvious is that the complement of $S(\tilde{T})$ relative to the order-n diamond must be a union of odd blocks. One way to see this is to tile the complement of the order-$(n-1)$ Aztec diamond in the fashion of Figure 3, obtaining an odd-deficient tiling $\tilde{T}^+$ of the entire plane. Then by the Lemma, $S(\tilde{T}^+)$ is an odd-deficient tiling of the plane; some of its missing odd-blocks lie in two semi-infinite strips of height 2 to the left and right of the order-n diamond, and all the others must lie strictly inside the order-n diamond. None of these blocks cross the boundary of the order-n diamond, so if we add these blocks to $S(\tilde{T})$, we get a complete tiling of the order-n diamond.

Consider now an odd-deficient tiling $\tilde{T}$ of the order-$(n-1)$ Aztec diamond, with $\tilde{T}_{\mathrm{vert}}$ equal to the set of vertical tiles of $\tilde{T}$; let

$$v(\tilde{T}) = \sum_{d \in \tilde{T}} v(d) = \sum_{d \in \tilde{T}_{\mathrm{vert}}} v(d)$$

and

$$r(\tilde{T}) = \sum_{d \in \tilde{T}} r(d) = \sum_{d \in \tilde{T}_{\mathrm{vert}}} r(d),$$

(recall that $v(d) = r(d) = 0$ for all horizontal dominoes d). Let

$$\mathrm{AD}(n-1, \tilde{T}; x, q) = \sum x^{v(T)} q^{r(T)},$$

where the sum is over all tilings T that extend $\tilde{T}$; we have

$$\mathrm{AD}(n-1;x,q) = \sum_{\tilde{T}} \mathrm{AD}(n-1,\tilde{T};x,q),$$

where the sum is over all partial tilings $\tilde{T}$ of the order-$(n-1)$ Aztec diamond. Say that $\tilde{T}$ is missing m odd blocks, so that it gives rise to 2^m distinct complete tilings T; then it is easily seen that

$$\mathrm{AD}(n-1,\tilde{T};x,q) = (1+xq^{-1})^m \prod_{d\in\tilde{T}} x^{v(d)}q^{r(d)} \ . \tag{8}$$

$S(\tilde{T})$ is an odd-deficient tiling of the order-n Aztec diamond with its odd coloring, missing $m+n$ odd blocks. Therefore, relative to the even coloring, we have

$$\mathrm{AD}(n,S(\tilde{T});x,q) = (1+xq)^{m+n} \prod_{d\in\tilde{T}_{\mathrm{vert}}} x^{v(S(d))}q^{-r(S(d))}.$$

The product in the right hand side can be rewritten as

$$\begin{aligned}\prod_{d\in\tilde{T}_{\mathrm{vert}}} x^{v(S(d))}q^{-r(S(d))} &= \prod_{d\in\tilde{T}_{\mathrm{vert}}} x^{v(d)}q^{r(d)+1} \\ &= \prod_{d\in\tilde{T}_{\mathrm{vert}}} (xq^2)^{v(d)}q^{r(d)} \\ &= \prod_{d\in\tilde{T}} (xq^2)^{v(d)}q^{r(d)}.\end{aligned}$$

But substituting n for $n-1$ and xq^2 for x in (8) yields

$$\mathrm{AD}(n,\tilde{T};xq^2,q) = (1+xq)^m \prod_{d\in\tilde{T}} (xq^2)^{v(d)}q^{r(d)}.$$

Hence

$$\mathrm{AD}(n,S(\tilde{T});x,q) = (1+xq)^n\,\mathrm{AD}(n-1,\tilde{T};xq^2,q).$$

Since every odd-deficient tiling of the order-n Aztec diamond with odd coloring is of the form $S(\tilde{T})$ for some odd-deficient tiling of the order-$(n-1)$ Aztec diamond with even coloring, we can sum both sides of the preceding equation over all $\tilde{T}$, obtaining

$$\mathrm{AD}(n;x,q) = (1+xq)^n\,\mathrm{AD}(n-1;xq^2,q).$$

The general formula for $\mathrm{AD}(n; x, q)$ follows immediately by induction.

Although this proof made no mention of alternating sign matrices, they are very much involved in determining the exact locations of the various 2-by-2 blocks. Specifically, let T be a domino tiling of the Aztec diamond of order $n-1$, and let A be the $(n-1)$-by-$(n-1)$ alternating sign matrix determined by T as in section 3. Then the locations of the odd blocks in $\tilde{T}$ are given by the 1's in A, while the locations of the odd blocks in $S(\tilde{T})$ are given by the -1's.

Latent within the proof of the formula for $\mathrm{AD}(n; x, q)$ is an iterative bijection between domino-tilings of the order-n Aztec diamond and bit-strings of length $n(n+1)/2$. Say we are given a bit-string of length $1+2+...+n$, and suppose we have already used the first $1+2+...+(k-1)$ bits to construct a domino-tiling of the order-$(k-1)$ diamond. Impose the even coloring on this Aztec diamond and locate the odd blocks, of which there are m. Pick up these odd blocks in some definite order (of which we will say more shortly) and put them elsewhere, retaining their order. Shuffle the dominoes in the remaining partial tiling of the Aztec diamond of order $k-1$. The resulting partial tiling of the order-k Aztec diamond has $m+k$ holes in it; fill these holes (again in some definite order) with the m blocks that were removed before, followed by k other blocks, whose orientations (horizontal vs. vertical) are determined by the next k bits of the bit-string. In this way one obtains a complete tiling of the Aztec diamond of order k. Note that no information has been lost; the procedure is fully reversible. Thus, iteration of the process gives a bijection between bit-strings of length $n(n+1)/2$ and domino tilings of the order-n Aztec diamond. Moreover, every 0 (resp. 1) in the bit-string leads to the creation of two horizontal (resp. vertical) dominoes in the tiling, so it is immediate that the number of tilings of the Aztec diamond with $2v$ vertical dominoes is $\binom{n(n+1)/2}{v}$.

The preceding construction requires a pairing between the m missing odd blocks of an odd-deficient tiling of the order-$(k-1)$ Aztec diamond and m of the $m+k$ missing odd blocks of an odd-deficient tiling of the order-n Aztec diamond. There is a canonical way of doing this pairing. Recall that these two kinds of blocks correspond to the -1's and $+1$'s in an alternating sign matrix A, so it suffices to decree some sort of pairing between the -1's and a subset of the $+1$'s (which will leave n $+1$'s left over). But this is easy: just pair each -1 with the next $+1$ below it in its column. In terms of shuffling, this means that the odd blocks of $\tilde{T}$ drift southeast until they find a hole in

$S(\tilde{T})$ that they can fit; this leaves n holes near the upper left border of the order-n Aztec diamond, which the n new 2-by-2 blocks exactly fill.

It would be nice to have a "shuffling" proof of the general formula (7) proved in section 4, and/or a procedure for randomly generating monotone triangles according to the (uneven) probability distribution given by the weights $W(\cdot)$.

7 Square ice

It is worthwhile to point out a connection between the combinatorial objects investigated in this paper and a statistical mechanical model that has been studied extensively since the 1960's. Recall that an n-by-n alternating sign matrix can be represented by its skewed summation, as in Figure 18(a). Replace each entry in the matrix by a node, and put a directed edge between every two adjacent entries, pointing from the smaller to the larger. Then one has a directed graph in which the circulation around every square cell is 0 (that is, each cell has two clockwise edges and two counterclockwise edges); see Figure 18(b). Finally, rotate each of these edges 90° counterclockwise about its midpoint. The end result is a configuration like the one shown in Figure 18(c), with divergence 0 at each node (that is, each node has two incoming arrows and two outgoing arrows). This is exactly the square-ice model of statistical mechanics, with the special boundary condition of incoming arrows along the left and right sides, and outgoing arrows along the top and bottom. (For discussion of this and related models, see [**1**] and [**12**].)

In the general square ice model, one associates a Boltzmann weight ω_i ($i = 1$ to 6) with each of the six possible vertex-configurations shown in Figure 19; then the weight of a configuration is defined as ${\omega_1}^{k_1}{\omega_2}^{k_2}...{\omega_6}^{k_6}$, where k_i is the number of vertices in the lattice of type i, and the **partition function** associated with the model (denoted by Z) is the sum of the weights of all possible configurations. Z has an implicit dependence on the lattice-size and the boundary conditions. It is customary to impose periodic boundary conditions, but we instead impose the "in-at-the-sides, out-at-the-top-and-bottom" condition on our n-by-n grid. Call this the **Aztec boundary condition**.

To recast our work on domino tilings of the Aztec diamond in terms of square ice, it is convenient to rephrase domino-tilings as dimer arrangements, or 1-factors. Specifically, we define a graph G' whose vertices correspond to the cells of the order-n Aztec diamond, with an edge between two vertices of G' if and only if the corresponding cells are adjacent. Then a domino-tiling of the Aztec diamond corresponds to a 1-factor F of G' (a collection of disjoint edges covering all vertices).

There is a general method for writing the number of 1-factors of a planar graph as a Pfaffian ([**7**]). Indeed, if one assigns weight $w(e)$ to each edge e of a planar graph on N vertices and defines the weight of a 1-factor as the product of its constituent weights, then the sum of the weights of all 1-factors of the graph is equal to the Pfaffian of an antisymmetric N-by-N matrix whose i, jth entry is $\pm w(e)$ if the graph has an edge e between i and j and 0 otherwise. (The delicate point is the correct choice of signs.) This method has been applied to the problem of counting 1-factors of m-by-n grids (equivalently, domino-tiling of m-by-n rectangles); see [**6**], [**2**], [**9**]. The Pfaffian method provides a yet another route to our result on tilings of the Aztec diamond, though we have omit the calculation here; see [**20**].

It is convenient to rotate the graph G' 45° clockwise, as in Figure 20(a). Call a cell of G' even or odd according to the parity of the corresponding vertex of G (under the standard coloring), so that the four extreme cells of G' are even. Every even cell is bounded by four edges, of which two, one, or none may be present in any particular 1-factor; the seven possibilities appear at the top of Figure 21, where a bold marking indicates the presence of an edge. If we replace each even cell by the corresponding ice-junction given at the bottom of the Figure 21, it is easy to check that the result is a valid ice-configuration satisfying our special boundary conditions, and that every such configuration arises in this way. The process is exemplified in Figure 20(b). Note that the transformation from 1-factors to ice-configurations is not one-to-one; it is in fact 2^{k_5}-to-one, where k_5 is the number of vertices of type 5 in the ice pattern. It can be checked that this transformation is equivalent to the more roundabout operation of converting the 1-factor to a domino-tiling, using the heights of the even vertices to form an n-by-n alternating sign matrix, and then turning the matrix into an ice-pattern as in the first paragraph of this section.

Let T be a tiling of the Aztec diamond, and let F be the associated 1-factor of G'. Note that every domino in T corresponds to an edge in F, and that this edge belongs to a unique even cell of G'. Hence, if we assign the weights x, x, 1, 1, 1, x^2, and 1 to the respective cell-figures, the product of the weights of the cell-figures appearing in F is equal to $x^{2v(T)}$. Thus, if we put

$$\begin{aligned} \omega_1 &= x, \\ \omega_2 &= x, \\ \omega_3 &= 1, \\ \omega_4 &= 1, \\ \omega_5 &= 1+x^2, \text{ and} \\ \omega_6 &= 1, \end{aligned}$$

then the partition function Z coincides with the generating function

$$\mathrm{AD}(n; x^2) = (1+x^2)^{n(n+1)/2}.$$

Note that $k_5 - k_6 = n$ for all order-n ice-configurations with Aztec boundary condition (corresponding to the fact that the number of 1's in an n-by-n alternating sign matrix must be n more than the number of -1's). Hence replacing ω_5 and ω_6 by $\sqrt{1+x^2}$ merely divides the partition function by $(1+x^2)^{n/2}$. Furthermore, $k_1 + k_2 + \ldots + k_6 = n^2$, so multiplying all the Boltzmann weights by a factor b merely multiplies the partition function by b^{n^2}. Writing $a = bx$ and $c = b\sqrt{x^2+1} = \sqrt{a^2+b^2}$, we see (after an easy calculation) that for the square-ice model with Aztec boundary condition and with Boltzmann weights

$$\begin{aligned} \omega_1 &= a, \\ \omega_2 &= a, \\ \omega_3 &= b, \\ \omega_4 &= b, \\ \omega_5 &= c, \\ \omega_6 &= c \end{aligned}$$

satisfying $a^2 + b^2 = c^2$, the partition function is given by $Z = c^{n^2}$.

It should be noticed that this family of special cases of the ice model (given by a, b, c satisfying $a^2 + b^2 = c^2$) is also the family that corresponds to the free fermion case, and is precisely the case in which the model has been solved by the method of Pfaffians. (See p. 151, 270-271 of [**1**], as well as [**4**].) This leads us to suspect that domino shuffling may in fact arise from some combinatorial interpretation of the Pfaffian solution.

We must emphasize the role played by the Aztec boundary conditions in the foregoing analysis, since it adds an element essentially foreign to the physical significance of the ice model. In particular, Lieb's solution of the ice model in the case $\omega_1 = \omega_2 = \ldots = \omega_6 = 1$ [**8**] tells us that there are asymptotically

$$\sqrt{64/27}^{\,n^2}$$

order-n ice-configurations with periodic boundary conditions; on the other hand, if the conjecture of Mills, Robbins, and Rumsey is correct, the number of order-n ice-configurations with Aztec boundary conditions should asymptotically be only

$$\sqrt{27/16}^{\,n^2}.$$

Clearly there are more constraints on a domino tiling near the boundary of an Aztec diamond than there are near the middle; this accounts for at least some of the drop in entropy. It would be interesting to know in a more quantitative way how the entropy of a random tiling is spatially distributed throughout a large Aztec diamond.

8 Epilogue

There have been many combinatorial transformations in this article, so it may be useful to review them.

First, we have:

(1) tilings;

(2) height-functions associated with tilings; and

(3) the order ideals associated with those height-functions.

We saw how to go from 1 to 2 (Thurston's marking scheme), from 2 to 3 (see the construction of the poset P in Section 3), and from 3 back to 1 (the stacked cubes).

Then we have:

(4) alternating sign matrices;

(5) height-functions associated with alternating sign matrices;

(6) the order ideals associated with those height-functions;

(7) monotone triangles;

(8) states of the square ice model (or equivalently its dual).

We saw the correspondence between (4) and (5) and between (5) and (6) in Section 3, between (4) and (7) in Section 4, and between (5) and (8) in Section 7. Further correspondences can be made. For instance, to get from (4) to (8) directly, we replace a 1 in an alternating sign matrix by a vertex-configuration of type 5, a -1 by a by a vertex-configuration of type 6, and each 0 by the unique vertex-configuartion of type 1-4 which fits in the pattern (note that arrows "go straight through" configurations of type 1-4 without reversing).

Then there are the mappings between (1)-(3) and (4)-(8), under the correspondence between domino tilings and compatible pairs of alternating sign matrices. We saw in Section 3 how to pass between (2) and (5), and between (3) and (6). Other connections can be made, and the reader might find it instructive to try to establish them.

There are actually even more incarnations of alternating sign matrices than have been discussed here: 3-colorings of certain graphs (subject to boundary constraints), 2-factors of some related graphs, and tilings of various regions in the plane by shapes of two kinds. These other structures may be discussed in a future paper. Then there are other combinatorial objects which appear (but have not been proved) to be equinumerous with the alternating sign matrices, namely, descending plane partitions and self-complementary totally symmetric plane partitions. See [**14**] for details.

Richard Stanley has discovered that our two-variable generating function for tilings of the order-n Aztec diamond is actually a specialization of a $2n$-variable generating function. A proof of this identity via the shuffling method of Section 6 is described in [**20**].

We wish to thank Carol Sandstrom and Chris Small for suggesting some of the terminology used in this article.

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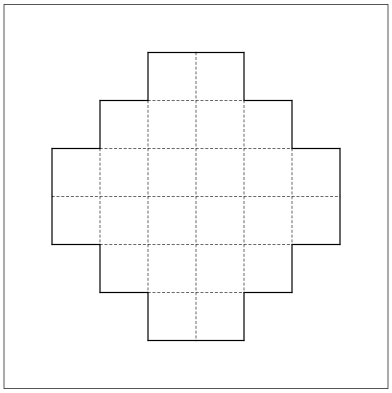


Figure 1

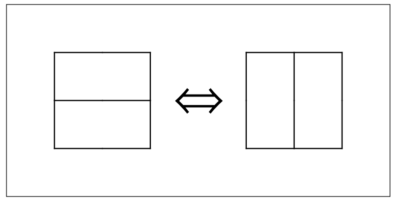
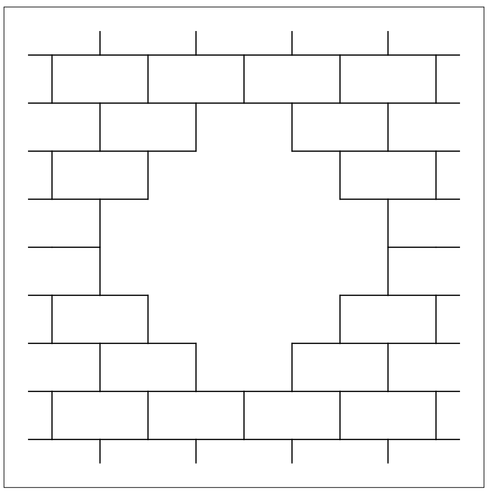


Figure 2



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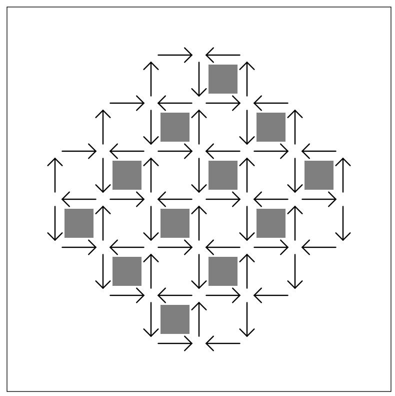

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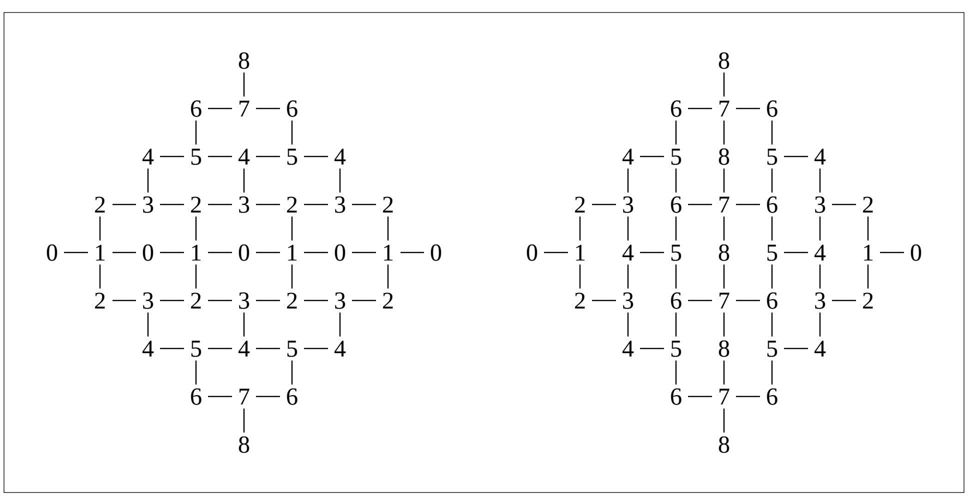
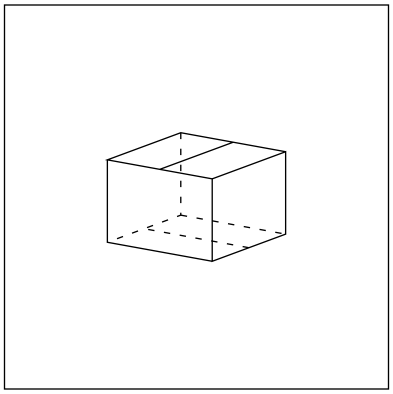


Figure 5



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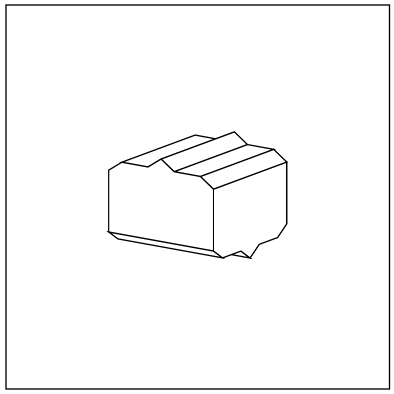
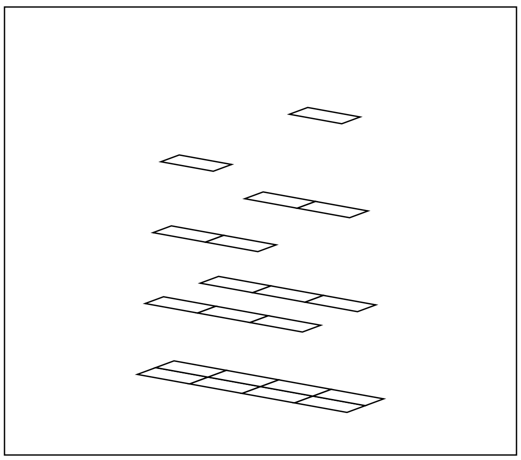


Figure 9



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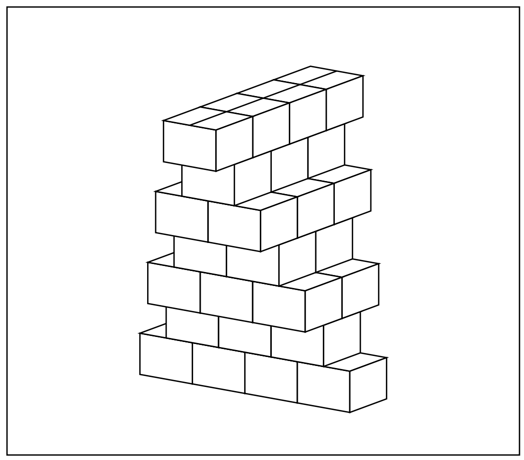

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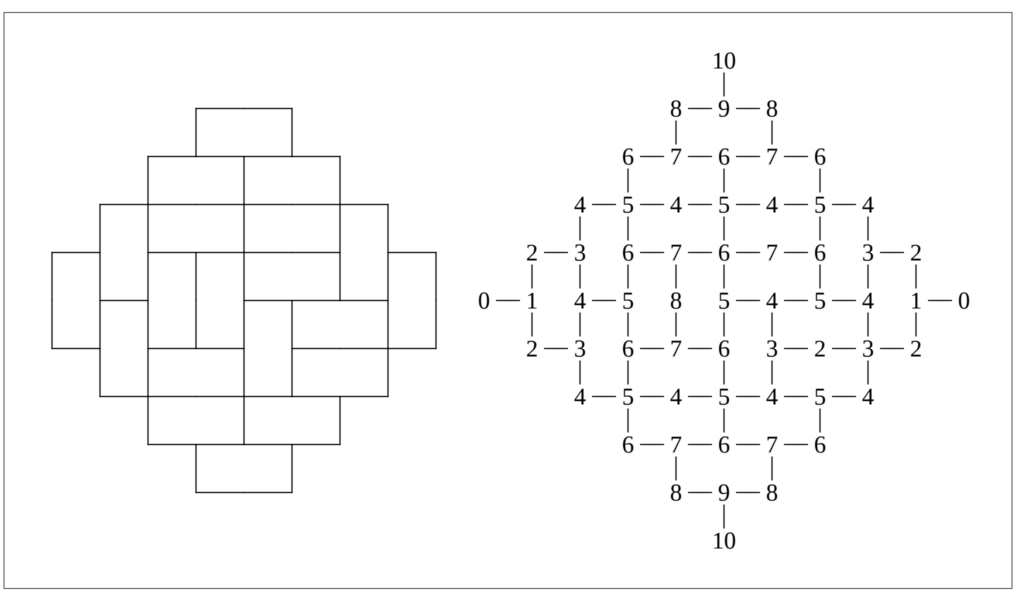
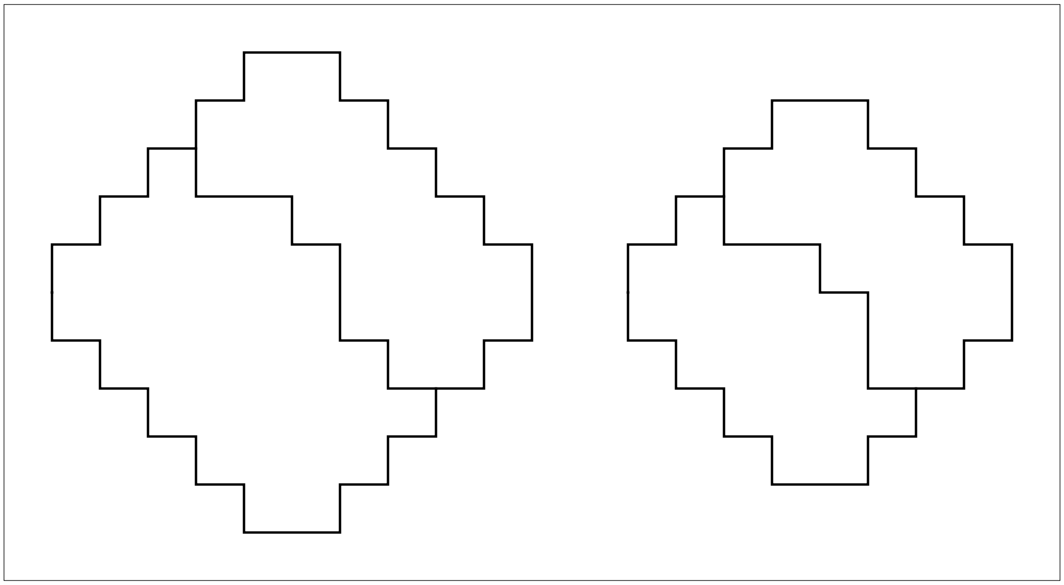


Figure 6



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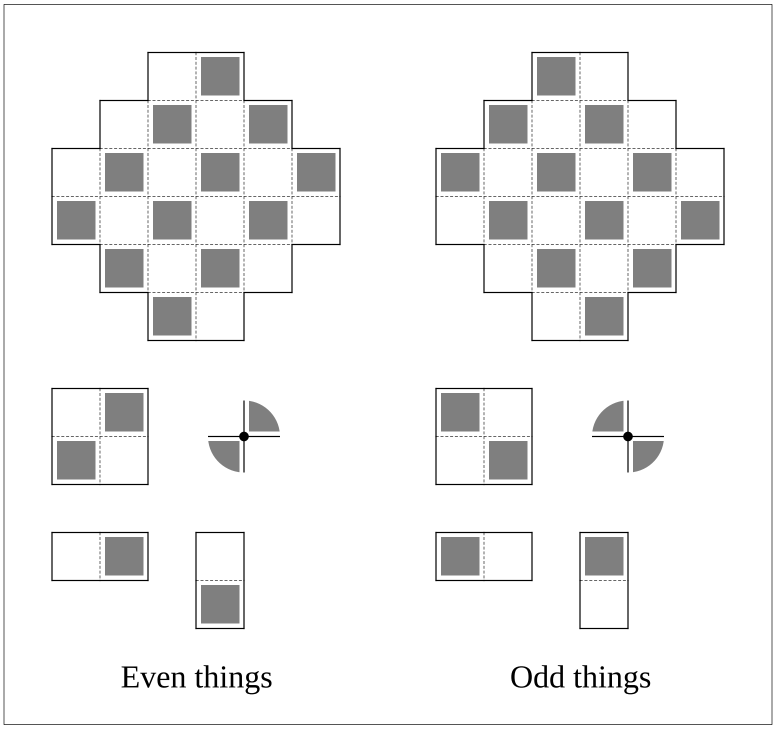

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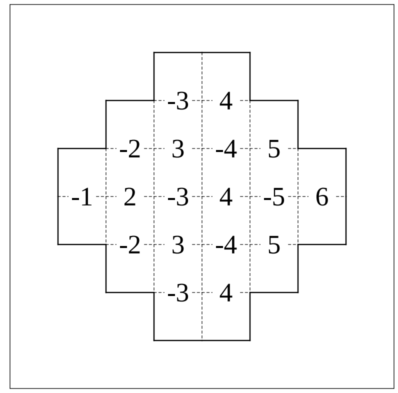

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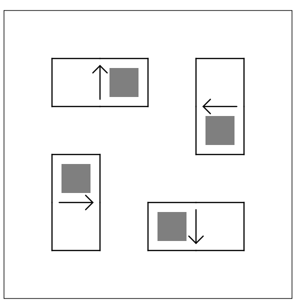

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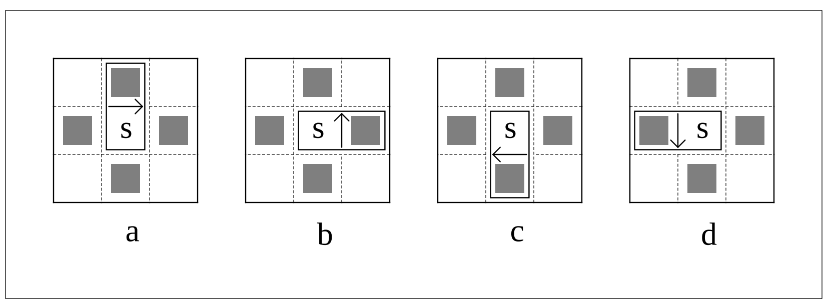
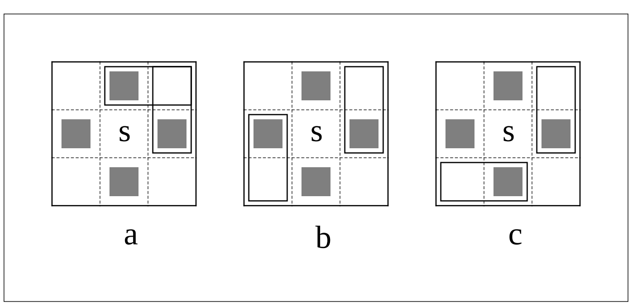


Figure 16



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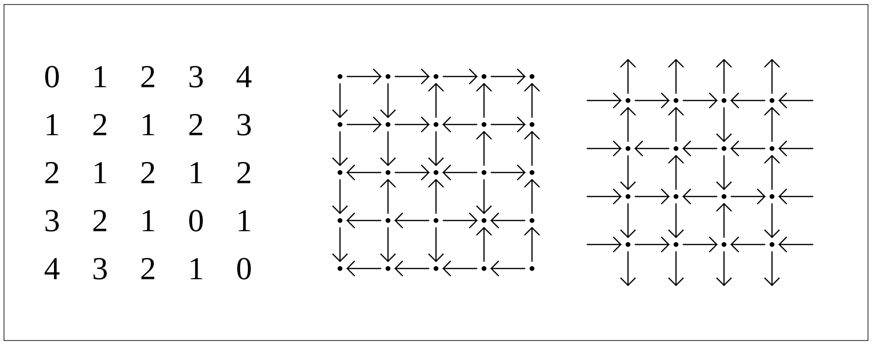

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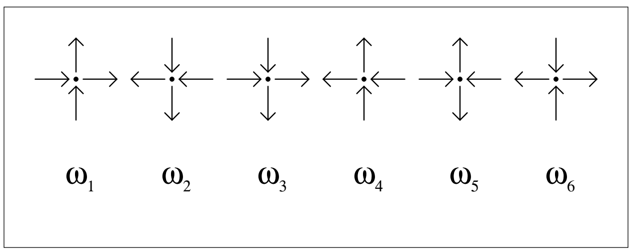

Figure 19

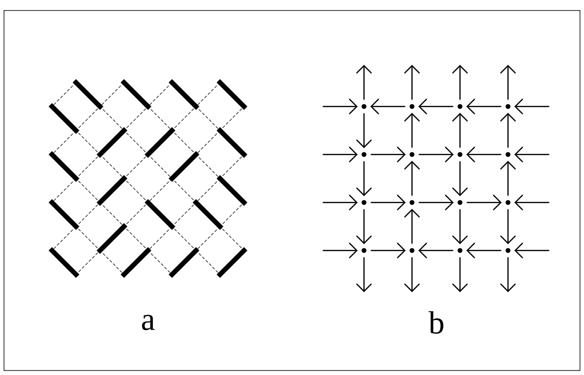

Figure 20

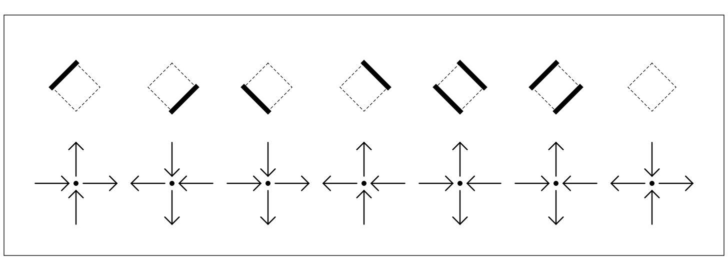

Figure 21