De-randomizing randomness with rotor-routers

by Jim Propp (UMass Lowell; visiting UC Berkeley and MSRI)

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These slides are on-line at http://jamespropp.org/bamc12.pdf so there's no need to take notes on anything you see here (only on the things that I say that you don't see!).

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Show that the bug must eventually leave the system (either by leaving site 1 heading to the left, or by leaving site 5 heading to the right), and give a simple rule for predicting which of the two outcomes will happen.

An example

What does the bug do if the initial state of the lights is GRGRG?

1	2	3	4	5
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G		R		R	⟵	G		G
G		R	⟵	G		G		G
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G		R	←	G		G		G	
G	←	G		G		G		G	
R	→	G		G		G		G	
R		R	→	G		G		G	
R		R		R	→	G		G	
R		R		R		R	→	G	
R		R		R		R		R	→

The bug successively visits sites 3,4,3,2,1,2,3,4,5, exiting at the right.

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This contradiction proves that the bug must escape: either it will go left from site 1, "arriving at site 0", or it will go right from site 5, "arriving at site 6".

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But which way will the bug escape?

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Either way, the quantity defined above is invariant, until the bug hits "site 0" or "site 6" (exiting at the left or right).

Predicting the future

So, if the bug goes from 3 to 0 (that is, it leaves the system heading left), then number of green lights must increase by 3;

and if the bug goes from 3 to 6 (that is, it leaves the system heading right), then the number of green lights must decrease by 3.

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Conclusion: The bug must exit to the right if the green lights outnumber the red lights, and to the left if the red lights outnumber the green lights.

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If you add lots of bugs to the system, one at a time, half of them will exit the system to the left and half will exit to the right.

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Etc.

"Homework": Show that out of any n successive bugs that enter the system, k will end up at site n and $n-k$ will end up at site 0.

But...

"What does any of this have to do with probability?"

The gambler's ruin problem

A gambler enters a casino with k dollars.

She makes a sequence of 1 dollar fair bets, so that on any given bet she has

- a probability of $1/2$ of gaining a dollar
- and a probability of $1/2$ of losing a dollar.

If she reaches her goal of n dollars, she leaves the casino happy; if she goes broke (ending up with 0 dollars), she leaves the casino unhappy.

It can be shown that the probability that she'll achieve her goal is k/n.

The gambler and the drunkard

The rising and falling fortunes of the gambler resemble the aimless steps of a drunkard.

Imagine an east-west street with buildings numbered 0 through n; building 0 (at the west end) is a police station, building n (at the east end) is a hotel, and building k is a bar.

A hotel-guest who has gone to the bar and gotten drunk leaves the bar and starts to wander.

- If he is in front of his hotel, the doorman will guide him inside;
- if he is in front of the police station, an officer will guide him to a cell;
- and if he is anywhere else, he makes a random choice of whether to head eastward or westward.

The drunkard's chance of getting to his hotel

It can be shown that the probability that the drunkard will reach his hotel is k/n.

Indeed, mathematically, there's no difference between the gambler and the drunkard.

If we have M drunkards successively leaving the bar, on average we expect $(k/n)M$ of them to get to the hotel (and the rest of them to end up in jail).

But this is just a statistical average, and our observations would be subject to statistical fluctuations, on the order of $\sqrt{M}$.

Drunkards vs. bugs

On the other hand, if we have M **bugs** successively leaving site k and following the colored-lights rule, the number of bugs that reach site n (rather than site 0) will also be close to $(k/n)M$; indeed, it will differ from $(k/n)M$ by at most n, regardless of how big M is. Note that this difference, n, is a lot smaller than $\sqrt{M}$ when M is big.

Randomness vs. quasirandomness

The drunkards make random decisions about where to go next; the decisions follow no pattern that would allow an observer to predict what will happen next.
The Law of Large Numbers says that with high probability, drunkards arriving at building i proceed to building $i-1$ about half the time and proceed to building $i+1$ about half the time.

The bugs make completely non-random decisions about where to go next. The rule that the bugs follow ensure that bugs visiting site i proceed to site $i-1$ half the time and proceed to site $i+1$ half the time.

The big lesson of quasirandom processes is that for many purposes, what matters is the half-half split (or the two-thirds-one-third split, or whatever it is), not where the split "comes from" (random choices versus simple rules).

More puzzles of this kind

Another puzzle of this kind is the Bugs on a Line problem. We'll work on this puzzle after the break.

Yet another is the Goldbug problem. We'll work on this one too, if time permits.

Like the five lights puzzle, these puzzles illustrate the way in which "quasirandom walk" mimics properties of random walk.

Rotor-routers

The building-blocks for quasirandom processes are called *rotor-routers*.

A k-way rotor-router at a site is a light that cycles through some fixed set of k colors, and sends each successive bug that visits the site to a neighboring site that is determined solely by the color of the light.

Machines built out of rotor-routers are *deterministic*: their behavior does not involve any element of chance.

But they have properties similar to those of their random counterparts.

Two-dimensional walk: random routing vs. rotor-routing

If a walker in an infinite square grid starts at (0,0) and repeatedly takes a random step to one of the four neighbors of its current location, the chance that the walker will reach (1,1) without returning to (0,0) can be shown to be $\pi/8$.

If you run rotor-router applet (designed and coded by U. Wisconsin undergraduates Hal Canary and Yutai Wong) and set the Graph/Mode selector to "2-D Walk", you'll see a rotor-router counterpart of the random walk process.

It was shown by Holroyd and P. that, as n goes to infinity, the proportion of the first n rotor-router walkers that reach (1,1) without returning to (0,0) converges to $\pi/8$.

In fact, the observed convergence to $\pi/8$ is faster than we can currently explain. (We can prove that the difference shrinks to 0 like $(\log n)/n$, but empirically it looks more like constant$/n$.)

Quasirandom and random blobs

Set the Graph/Mode selector to "2-D Aggregation" to see a quasirandom gadget for growing blobs of bugs. When a bug arrives at a vacant site, it stays there forever; when a bug arrives at an occupied site, it uses a rotor at that site to tell it where to go next.

In the corresponding random growth process (called Internal Diffusion-Limited Aggregation or "IDLA"), a bug that arrives at an occupied site moves randomly to one of the neighboring site. Lawler, Bramson, and Griffeath showed that over time, the shape of the growing blob converges to a circle.

To see what the random growth process looks like, visit http://www.wisdom.weizmann.ac.il/~itsik/Rw/Simulation.html.

Randomness vs. roundness

The quasirandom growth process grows blobs that empirically are even rounder than the random growth process, but nobody has proved this rigorously.

Lionel Levine and Yuval Peres proved in 2005 that the quasirandom blobs really do become true circles in the limit.

http://jamespropp.org/million.gif shows what the quasirandom blob looks like after the blob has grown to size one million. The internal structures are still completely mysterious.

Clever simulation

Tobias Friedrich and Lionel Levine devised a clever scheme for finding out what the blob of size n is that doesn't require directly simulating all the moves that the bugs would follow.

Their method has allowed them to compute the rotor-router blob of size one *billion*.

Friedrich's webpage http://rotor-router.mpi-inf.mpg.de/ shows rotor-router blobs of various kinds and sizes, using a Google-maps navigational interface.

The future

There are many other examples of simple quasirandom processes that exhibit strange patterns that we do not understand at all.

I expect it will take decades before rigorous mathematical theory catches up with computer-assisted mathematical exploration.

Back to some puzzles!

In the last hour, we'll look at two rotor-router puzzles.

But first, let's talk about the origins of the puzzles, in the theory of biased 1-dimensional random walk.

In the first kind of random walk we'll look at, a bug is more likely to jump to the right than to the left.

In the second kind of random walk, the bug is equally likely to jump either way, but it takes bigger steps when it moves to the left.

Biased walk #1

A bug is placed at 1.

If the bug ever reaches 0, the game is over.

At each time step, the bug jumps 1 step to the left with probablity 1/3 and 1 step to the right with probability 2/3.

What is the chance that the bug ever reaches 0?

Solution

Let p be the probability that a bug started at 1 ever reaches 0.

Let q be the probability that a bug started at 2 ever reaches 1.

Let r be the probability that a bug started at 2 ever reaches 0.

What are some relations among these three probabilities?

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So $p = (1/3)(1) + (2/3)p^2$, which has two roots: $p = 1$ and $p = 1/2$. It can be shown that the root $p = 1$ is extraneous, and that the correct answer is $1/2$.

The Bugs on a Line puzzle

Each positive integer on the number line is equipped with a green, yellow or red light. A bug is dropped on 1 and obeys the following rules at all times: if it sees a green light, it turns the light yellow and moves one step to the right; if it sees a yellow light, it turns the light red and moves one step to the right; if it sees a red light, it turns the light green and moves one step to the left.

Eventually the bug will fall off the line to the left, or run out to infinity on the right. A second bug is then dropped on 1, then a third.

Assume that the lights are not initially all green or all red.

Prove that if the first bug falls off to the left, the second will march off to infinity on the right, and vice versa.

Trick

Think of a green light as the digit 0, red as the digit 1, and yellow as the "digit" 1/2. The configuration of lights can then be thought of as a number between 0 and 1 written out in binary,

$$x = .x_1 x_2 x_3 \ldots$$

where, numerically,

$$x = x_1 \cdot (1/2)^1 + x_2 \cdot (1/2)^2 + \cdots \ .$$

This is the "value" of the lights.

Think of the bug itself as having value $(1/2)^i$ when it is in position i.

Then the value of the lights plus the value of the bug is invariant, that is, it does not change as the bug moves.

The upshot

When the bug you add at 1 exits at 0, the net change in the value of the lights is $-1/2$.

When the bug you add at 1 wanders off to infinity, the net change in the value of the lights is $+1/2$.

But the total value of the lights must always be between $.000\cdots = 0$ and $.111\cdots = 1$.

Moreover, the initial value of the system can't be 0 or 1, because the lights aren't all 0's or all 1's.

So we can't have both bugs exit at 0 (because the net change in the value of the lights can't be -1), and we can't have both bugs wander off to infinity (because the net change in the value of the lights can't be $+1$).

So the bugs must have opposite destinies.

Biased walk #2

A bug is placed at 1.

If the bug ever reaches 0 or -1, the game is over.

At each time step, the bug jumps 2 steps to the left with probablity $1/2$ and 1 step to the right with probability $1/2$.

It can be shown that the bug is certain (probability 1) of eventually reaching either 0 or -1.

What is the chance that the bug ever reaches -1?

Solution

Let p be the probability that a bug started at 1 ever reaches -1.

Let q be the probability that a bug started at 2 ever reaches 0.

Let r be the probability that a bug started at 2 ever reaches -1.

What are some relations among these three probabilities?

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$r = p(1 - q)$ (the bug can only get from 2 to -1 via 1)

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$p = (1/2)(1) + (1/2)(r)$ (start a bug at 1 and look at where the bug is after its first jump)

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So $p = (1/2)(1) + (1/2)p(1 - p)$, which has roots $p = (-1 \pm \sqrt{5})/2$. The negative root is clearly extraneous, so the correct answer is $p = (-1 + \sqrt{5})/2 = 0.618\ldots$.

Note that $1 - p = p^2$, so $p/(1 - p) = 1/p =$ the golden ratio.

The Goldbug puzzle

Each positive integer on the number line is equipped with a blue or yellow light. All lights are initially blue. A bug is dropped on 1 and obeys the following rules at all times: if it sees a yellow light, it turns the light blue and moves one step to the right; if it sees a blue light, it turns the light yellow and moves TWO steps to the left.

Eventually the bug will fall off the line to the left, landing at either -1 or 0. A second bug is then dropped on 1, then a third, and so on. Each successive bug that is added falls off the line to the left, landing at either -1 or 0. (Prove this!)

Show that the number of bugs that land at -1, divided by the number of bugs that land at 0, converges to $\Phi = (1 + \sqrt{5})/2 = 1.618...$, the "golden ratio".

Trick

Once again, you construct an invariant; this is not built on base 1 (like Puzzle #1) or base 2 (like Puzzle #2), but on "base Φ" (or, alternatively, "base Fibonacci").

One fun way to see how Fibonacci numbers play a role is to assign blue rotors the value 0 when they're blue, and to assign yellow rotors the respective values 1, 2, 3, 5, 8, ...

Give the bug the value 1.

Then, after the nth bug has passed through the system, the yellow rotors have Fibonacci values that sum to n!

This way of assigning values to the rotors and the bug gives a way to prove that each bug eventually lands at either -1 or 0.

For more information

You can read more about the Bugs on a Line problem in Peter Winkler's book *Mathematical Puzzles: A Connoissuers Collection*. (See the Bugs on a Line problem on page 82, with solution on pages 91–93.)

You can read more about the Goldbug problem in Michael Kleber's article "Goldbug Variations", published in the Winter 2005 issue of *The Mathematical Intelligencer* and also available on the web at http://arxiv.org/abs/math/0501497.

These slides are at http://jamespropp.org/bamc12.pdf.