Bugs, Blobs, and Rotor-Routers: What happens to probability theory when you get rid of randomness?

by Jim Propp (UMass Lowell; visiting UC Berkeley and MSRI)

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These slides are on-line at http://jamespropp.org/bamo12.pdf so there's no need to take notes on anything you see here (only on the things that I say that you don't see!).

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Show that the bug must eventually leave the system (either by leaving site 1 heading to the left, or by leaving site 5 heading to the right), and give a simple rule for predicting which of the two outcomes will happen.

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The bug successively visits sites 3,4,3,2,1,2,3,4,5, exiting at the right.

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This contradiction proves that the bug must escape: either it will go left from site 1, "arriving at site 0", or it will go right from site 5, "arriving at site 6".

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Either way, the quantity defined above is invariant, until the bug hits "site 0" or "site 6" (exiting at the left or right).

Predicting the future

So, if the bug goes from 3 to 0 (that is, it leaves the system heading left), then number of green lights must increase by 3;

and if the bug goes from 3 to 6 (that is, it leaves the system heading right), then the number of green lights must decrease by 3.

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Conclusion: The bug must exit to the right if the green lights outnumber the red lights, and to the left if the red lights outnumber the green lights.

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If you add lots of bugs to the system, one at a time, half of them will exit the system to the left and half will exit to the right.

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"Homework": Show that out of any n successive bugs that enter the system, k will end up at site n and $n-k$ will end up at site 0.

But...

“What does any of this have to do with probability?”

The gambler's ruin problem

A gambler enters a casino with k dollars.

She makes a sequence of 1 dollar fair bets, so that on any given bet she has

- a probability of $1/2$ of gaining a dollar
- and a probability of $1/2$ of losing a dollar.

If she reaches her goal of n dollars, she leaves the casino happy; if she goes broke (ending up with 0 dollars), she leaves the casino unhappy.

It can be shown that the probability that she'll achieve her goal is k/n.

The gambler and the drunkard

The rising and falling fortunes of the gambler resemble the aimless steps of a drunkard.

Imagine an east-west street with buildings numbered 0 through n; building 0 (at the west end) is a police station, building n (at the east end) is a hotel, and building k is a bar.

A hotel-guest who has gone to the bar and gotten drunk leaves the bar and starts to wander.

- If he is in front of his hotel, the doorman will guide him inside;
- if he is in front of the police station, an officer will guide him to a cell;
- and if he is anywhere else, he makes a random choice of whether to head eastward or westward.

The drunkard's chance of getting to his hotel

It can be shown that the probability that the drunkard will reach his hotel is k/n.

Indeed, mathematically, there's no difference between the gambler and the drunkard.

If we have M drunkards successively leaving the bar, on average we expect $(k/n)M$ of them to get to the hotel (and the rest of them to end up in jail).

But this is just a statistical average, and our observations would be subject to statistical fluctuations, on the order of $\sqrt{M}$.

Drunkards vs. bugs

On the other hand, if we have M **bugs** successively leaving site k and following the colored-lights rule, the number of bugs that reach site n (rather than site 0) will also be close to $(k/n)M$; indeed, it will differ from $(k/n)M$ by at most n, regardless of how big M is. Note that this difference, n, is a lot smaller than $\sqrt{M}$ when M is big.

Randomness vs. quasirandomness

The drunkards make random decisions about where to go next; the decisions follow no pattern that would allow an observer to predict what will happen next.
The Law of Large Numbers says that with high probability, drunkards arriving at building i proceed to building $i-1$ about half the time and proceed to building $i+1$ about half the time.

The bugs make completely non-random decisions about where to go next. The rule that the bugs follow ensure that bugs visiting site i proceed to site $i-1$ half the time and proceed to site $i+1$ half the time.

The big lesson of quasirandom processes is that for many purposes, what matters is the half-half split (or the two-thirds-one-third split, or whatever it is), not where the split "comes from" (random choices versus simple rules).

More puzzles of this kind

Another puzzle of this kind is the Bugs on a Line problem, published in Peter Winkler's book *Mathematical Puzzles: A Connoissuers Collection*. The problem appears on on page 82, and the solution appears on pages 91–93.

Yet another is the Goldbug problem, published in Michael Kleber's article "Goldbug Variations". This article appeared in the Winter 2005 issue of *The Mathematical Intelligencer*, and is available on the web at http://front.math.ucdavis.edu/0501.5497.

Like the five lights puzzle, these puzzles illustrate the way in which "quasirandom walk" mimics properties of random walk.

Rotor-routers

The building-blocks for quasirandom processes are called *rotor-routers*.

A k-way rotor-router at a site is a light that cycles through some fixed set of k colors, and sends each successive bug that visits the site to a neighboring site that is determined solely by the color of the light.

Machines built out of rotor-routers are *deterministic*: their behavior does not involve any element of chance.

But they have properties similar to those of their random counterparts.

Two-dimensional walk: random routing vs. rotor-routing

If a walker in an infinite square grid starts at (0,0) and repeatedly takes a random step to one of the four neighbors of its current location, the chance that the walker will reach (1,1) without returning to (0,0) can be shown to be $\pi/8$.

If you run rotor-router applet (designed and coded by U. Wisconsin undergraduates Hal Canary and Yutai Wong) and set the Graph/Mode selector to "2-D Walk", you'll see a rotor-router counterpart of the random walk process.

It was shown by Holroyd and P. that, as n goes to infinity, the proportion of the first n rotor-router walkers that reach (1,1) without returning to (0,0) converges to $\pi/8$.

In fact, the observed convergence to $\pi/8$ is faster than we can currently explain. (We can prove that the difference shrinks to 0 like $(\log n)/n$, but empirically it looks more like constant$/n$.)

Quasirandom and random blobs

Set the Graph/Mode selector to "2-D Aggregation" to see a quasirandom gadget for growing blobs of bugs. When a bug arrives at a vacant site, it stays there forever; when a bug arrives at an occupied site, it uses a rotor at that site to tell it where to go next.

In the corresponding random growth process (called Internal Diffusion-Limited Aggregation or "IDLA"), a bug that arrives at an occupied site moves randomly to one of the neighboring site. Lawler, Bramson, and Griffeath showed that over time, the shape of the growing blob converges to a circle.

To see what the random growth process looks like, visit http://www.wisdom.weizmann.ac.il/∼itsik/Rw/Simulation.html.

Randomness vs. roundness

The quasirandom growth process grows blobs that empirically are even rounder than the random growth process, but nobody has proved this rigorously.

Lionel Levine and Yuval Peres proved in 2005 that the quasirandom blobs really do become true circles in the limit.

http://jamespropp.org/million.gif shows what the quasirandom blob looks like after the blob has grown to size one million. The internal structures are still completely mysterious.

Clever simulation

Tobias Friedrich and Lionel Levine devised a clever scheme for finding out what the blob of size n is that doesn't require directly simulating all the moves that the bugs would follow.

Their method has allowed them to compute the rotor-router blob of size one *billion*.

Friedrich's webpage http://rotor-router.mpi-inf.mpg.de/ shows rotor-router blobs of various kinds and sizes, using a Google-maps navigational interface.

The future

There are many other examples of simple quasirandom processes that exhibit strange patterns that we do not understand at all.

I expect it will take decades before rigorous mathematical theory catches up with computer-assisted mathematical exploration.