Quasirandom Aggregation with Rotor-Routers

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(based on an article in progress with Lionel Levine;

with thanks also to Hal Canary, Matt Cook, Ander Holroyd, Dan Hoey, and Michael Kleber)

For more details, see:

`http://www.math.wisc.edu/~propp/hidden/rotor`

(a memo I wrote in 2002) and

arxiv: `math.CO/0409407`

("The Rotor-Router Model", by Lionel Levine); also try the applet

`http://www.math.wisc.edu/~propp/rotor-router-1.0/`

(written by Hal Canary and Francis Wong) and the web-stuff at

`http://www.paradise.caltech.edu/~cook/Warehouse/ForPropp/`

(written by Matt Cook).

Diffusion-limited aggregation

Diffusion-limited aggregation (DLA) is a (probabilistic) growth process in R^2.

For purposes of this talk, we will replace it by a discrete analogue that creates a sequence of finite subsets $S_1, S_2, \ldots$ of the lattice Z^d.

Here S_1 is the singleton consisting of the origin $\mathbf{0}$, and for all $n > 1$, S_n is equal to S_{n-1} with a single lattice-point x_n adjoined, where x_n is adjacent to some element of S_{n-1}.

To choose x_n, imagine a particle that wanders in "from infinity", and let x_n be the first site that the particle visits that is adjacent to S_{n-1}.

See "Diffusion-Limited Aggregation: A Model for Pattern Formation", http://www.aip.org/pt/vol-53/iss-11/p36.html (Physics Today On-Line).

Internal DLA

Internal DLA is a variant of DLA in which particles are added at the origin rather than at infinity.

Unlike external DLA, which gives rise to feathery dendritic shapes of a conjecturally fractal nature that have so far mostly defied rigorous analysis, internal DLA is mathematically tractable: Lawler, Bramson, and Griffeath (1992) proved that the set S_n, rescaled by $\sqrt{n}$, converges in probability to a disk.

See `http://www.santafe.edu/~moore/pubs/dla.html` for information on efficient computation of IDLA.

The Diaconis-Fulton model

Diaconis and Fulton (1991) define a way to add two finite subsets of Z^d (or any infinite connected graph G).

If the sets S and T are disjoint, their sum is just their union.

If the intersection of S and T consists of a single point x, the sum of S and T is $S \cup T$ together with a random point y obtained by letting a particle execute a random walk in G starting from x until it first lands on a point y not in $S \cup T$.

More generally, if S and T overlap in a set $\{x_1, ..., x_k\}$, the sum of S and T is a random set of cardinality $|S|+|T|$, constructed by letting k particles do random walk starting from the respective points $x_1, x_2, ..., x_k$, terminating in distinct points

$$\begin{aligned} &y_1 \notin S \cup T, \\ &y_2 \notin S \cup T \cup \{y_1\}, \\ &y_3 \notin S \cup T \cup \{y_1, y_2\}, \\ &... \\ &y_k \notin S \cup T \cup \{y_1, y_2, ..., y_{k-1}\} \end{aligned}$$

and adjoining these k new points.

This might not appear to be a well-defined definition of addition, since one might suppose that the probability law governing the resulting set

$$S \cup T \cup \{y_1, ..., y_k\}$$

would depend on the choice of ordering of the points x_i in the intersection of S and T.

However, it turns out that the ordering of the x_i's does not matter.

A good way to see this is to use a "stacks" picture.

Associate a stack with each vertex v.

The stack at v consists of an infinite sequences of vertices, each of which is a random neighbor of v; all the elements of all the stacks are independent of one another.

Then, to do a random walk from a vertex v, one pops the top element of the stack of v (call it v') and walks to v', then one pops the top element of *that* stack (call it v") and walks to v", etc., until the stopping-condition is satisfied (that is, the particle arrives at a vertex that is not already occupied).

Relative to the randomness embodied in the entries of the stacks, the Diaconis-Fulton operation is deterministic, and one need only prove that the outcome does not depend on the order in which one resolves conflicts (i.e. situations in which two particles start out attempting to occupy the same site).

Proving this claim can be reduced to the simpler claim that $(S + \{x\}) + \{y\}$ is the same set as $(S + \{y\}) + \{x\}$.

In the Diaconis-Fulton framework, internal DLA can be described very succinctly as the result of adding the singleton $\{\mathbf{0}\}$ to itself repeatedly.

(Diaconis-Fulton addition is really a way of adding two probability distributions on the collection of finite sets.)

IDLA in 1 dimension

Say the current blob is $[-x(t), y(t)] = [-x, y]$.

The probability that the next particle escapes at the left resp. right is $y/(x+y)$ resp. $x/(x+y)$ (gambler's ruin).

Continuum limit:

$$dx/dt = y/(x+y),\ dy/dt = x/(x+y)$$

$$dy/dx = x/y$$

$$y\ dy = x\ dx$$

$$y^2 = x^2 + C$$

$$x/y \to 1$$

Variant: Every time a particle escapes to the left, put 1 particle in each of the r sites immediately to the left of the blob. Every time a particle escapes to the right, put 1 particle in each of the s sites immediately to the right of the blob.

Continuum limit:

$dx/dt = ry/(x+y),\ dy/dt = sx/(x+y)$

$$dy/dx = sx/ry$$

$$ry\ dy = sx\ dx$$

$$ry^2 = sx^2 + C$$

$$x/y \to \sqrt{r/s}$$

Quasirandom IDLA in 1 dimension

In each stack, we replace *randomness* by *low discrepancy*.

Specifically, at each site, we alternate between ejecting particles to the left and ejecting particles to the right.

Let $-x_n$ and y_n be the leftmost and rightmost sites in the blob after n particles have been added.

Theorem (Levine): $|x_n\sqrt{s} - y_n\sqrt{r}|$ is bounded.

Key ideas:

- model dynamics as iteration of a piecewise linear map;
- find quadratic invariants.

Specifically, the piecewise linear map

$$f_{r,s}(x,y,z) = \begin{cases} (x-r, y, z-x+1) \\ \qquad \text{if } x+y>z, \\ (x, y+s, z-y) \\ \qquad \text{if } x+y \le z \end{cases}$$

implements the operation of adding a particle at $\mathbf{0}$ and letting it reach the boundary of the blob, and the quadratic quantity

$$sx^2 - ry^2 + (r-2)sx + rsy - 2rsz$$

is invariant under $f_{r,s}$.

Quasirandom IDLA in 2 dimensions

Recall:

Theorem (Bramson, Griffeath, and Lawler): The n-particle IDLA blob in two dimensions is round to within radial fluctuation that are almost surely $o(\sqrt{n})$.

Theorem (Lawler): Can replace $o(\sqrt{n})$ by $O(n^{1/3})$.

Theorem (Blachère): Can replace $O(n^{1/3})$ by $O(\ln n)$.

(Simulations by Moore and Machta suggest that this last result is best possible.)

To quasirandomize, put a 4-valued rotor at each site.

The rotor can point East, South, West, or North.

When a particle arrives at an already-occupied site, the rotor at the site rotates, and the particle moves in the direction of the new rotor-setting.

When a particle arrives at an unoccupied site, it stays there, and a new particle gets added to the system at the origin (and starts a walk of its own).

Each time a new particle gets added to the blob, the rotor-settings form a spanning forest rooted at the boundary of the blob.

How round is the circle made by the rotor-routers?

After a million particles have been added to the system, the inradius is 563.5 and the outradius is 565.1: they differ by 1.6 (about three tenths of one percent).

The **difference** between inradius and circumradius seems to be bounded.

No one has proved that **ratio** between inradius and circumradius is bounded.

Can Kleber's approach to proving that fluctuations are $o(\sqrt{n})$ (comparison with IDLA) be made to work?

If not, can we derandomize the Lawler, Bramson, and Griffeath argument?

Observed features of quasirandom aggregation in 2D

- Distribution of angles between successive ejections is heavily biased towards the left half of $[0, 2\pi)$, with a sharp peak at $\pi/2$.
- Monochromatic regions near the boundary appear at locations (x, y) with y/x close to a simple rational number.
- Monochromatic regions inside appear at locations of the form $(f(i), f(j))$ where i, j are integers and f is a simple conformal map of the plane (Hoey).

These structures offer some of the same features as blobs created by chip-firing, but are probably easier to understand.

Digression: Other things to try

- Directed IDLA (NENENE...)
- Other rules, e.g. ENWENWENW...
- (DLA and Directed DLA)

It would also be natural to look at aggregation on trees (cf. Aldous and Shields).

I have no idea how to quasirandomize Rost's growth model in a quadrant (which yields a parabolic boundary) or the discrete time version of Rost (which yields a circular boundary), but I'd very much like to; it might give a construction of quasirandom tilings.

Diaconis-Fulton in the continuum?

Conjecture: A continuum limit of Diaconis-Fulton addition exists and is rotationally invariant.

Simulation suggests that the continuum limit of rotor-router version of Diaconis-Fulton exists and is rotationally invariant (experiments by self and by Matt Cook).