



The Problem of the Calissons

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NOTES

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The Problem of the Calissons

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A *calisson* is a French sweet that looks like two equilateral triangles meeting along an edge. Calissons could come in a box shaped like a regular hexagon, and their packing would suggest an interesting combinatorial problem. Suppose a box with side of length n is filled with sweets of sides of length 1. The long diagonal of each calisson in the box is parallel to one of three different lines, as in the picture.

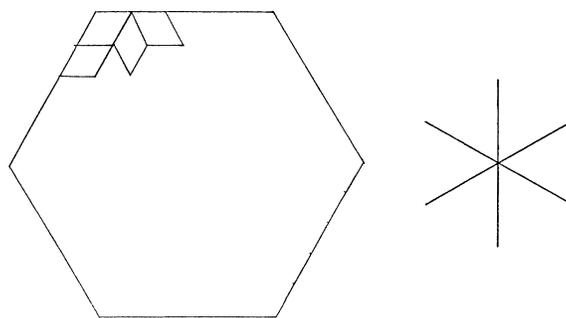


FIG. 1

We will refer to these three possibilities by saying that a calisson admits three distinct orientations. Our main result is the following: in any packing, the number of calissons with a given orientation is one third of the total number of calissons in the box.

C. Frederico Palmeira, of PUC-Rio, Brazil, heard of the problem from Jean Martinet (Université de Strasbourg), and showed it to one of the authors. We believe that the result is known and proven, but we were unable to find any references.

The idea of the proof is to reduce the problem to a very intuitive fact in three dimensions. We do not give complete details of a formal proof for two reasons. First, we do not want to spoil the simplicity of the basic intuitive idea. Furthermore, the result is an example of a proof by picture that does not translate immediately into precise mathematics. However, once some clues are given, we think it would be a nice exercise for an undergraduate class to fill in the details, by converting obvious geometric facts into precise mathematical statements.

We now sketch the proof. Draw an arbitrary filling of a box by calissons, such as the one below (in this case, n is 5).

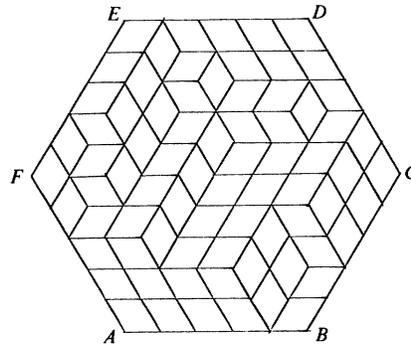


FIG. 2

If you rotate the picture counterclockwise a little bit, it is almost impossible not to ‘see’ a collection of cubes over a (square) floor, between two (square) walls.

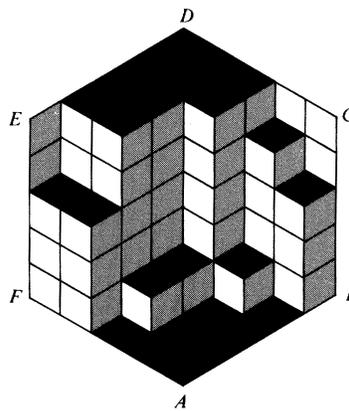


FIG. 3

If you were to look at this array of cubes from above, you would see exactly those faces with a certain orientation, and, of course, these faces will cover the floor. If you now look at a side wall, the same thing will happen—you will see all the faces with another fixed orientation covering the wall. So, the number of faces with a given orientation is the same for all three orientations, QED (informally)!

Let us give some idea of how a precise proof might go. First, it is not obvious that a filling of the hexagon corresponds to some arrangement of cubes within the walls and the floor. To prove this, we give a recipe for ‘lifting’ an arbitrary filling: we will assign coordinates to the vertices of the calissons as if they were points inside an $n \times n \times n$ cube in three-dimensional space. Give coordinates $(n, n, 0)$ to vertex A in Figure 3. Now, join an arbitrary vertex V to A by a path along the sides of the calissons. Starting from A , coordinatize the vertices along the path according

to the following rules. Suppose you reach vertex U by moving along a side of a calisson from vertex W with coordinates (a, b, c) . Then:

- a) if the side WU is parallel to AF in figure 3, give U coordinates $(a, b - 1, c)$ if U is higher than W in figure 3 and coordinates $(a, b + 1, c)$ otherwise;
- b) if WU is parallel to BC , give U coordinates $(a, b, c + 1)$ if U is higher than W and coordinates $(a, b, c - 1)$ otherwise;
- c) if WU is parallel to AB , give U coordinates $(a - 1, b, c)$ if U is higher than W and coordinates $(a + 1, b, c)$ otherwise.

We leave to the reader the task of proving that the coordinatization above is well defined.

To prove the remaining statements in the informal proof, we give a convenient definition. Let s be a side of a calisson along an edge of the box. An s -chain is a sequence of distinct calissons defined inductively as follows. The first calisson in the chain is the one having s as a side. The $(k + 1)$ st calisson meets the k th one at a side parallel to s . The edge s can be parallel to three different directions, d_1 , d_2 and d_3 . We say that an s -chain is of type d_i if s is parallel to d_i . The following facts are rather easy to prove, using the coordinatization defined above.

1. An s -chain goes from one side of the box to the other side parallel to s .
2. The long diagonal of a calisson in an s -chain is never perpendicular to s .
3. Chains of different types have a calisson in common; chains of the same type never meet.
4. All calissons that lie in two chains of different type have the same orientation.

There are n chains of each type. So, fixing two types, there are at least n^2 distinct calissons with a fixed orientation, from the facts above. As the total number of calissons is $3n^2$ (simply by dividing the area of the box by the area of one calisson), the number of calissons with a fixed orientation has to be n^2 , and the main result is proved.

As a final exercise, we invite the reader to show that the configuration of calissons drawn below cannot be completed to fill a box (hint: notice that n is 5, and one already has six cubes stacked one on top of the other).

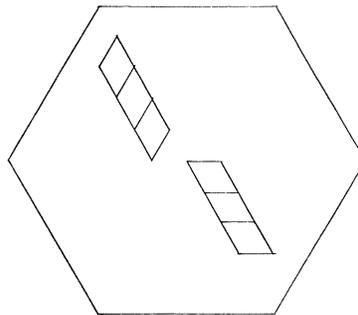


FIG. 4

The authors would like to thank the referees for suggesting many improvements in the presentation of the article.