Math 431, Assignment #1: Solutions

(due 2/1/01)

1. (a) If a student can receive any number of awards, then the best scholarship prize can go to any of the 30 students, the best leadership qualities prize can then go to *any* of the 30 students (including the one who won the best scholarship prize), and so on. By the basic principle of counting, the total number of ways to assign prizes is $30 \cdot 30 \cdot 30 \cdot 30 \cdot 30$, or 30^5, or 24,300,000. (b) If each student can receive at most one award, then the best scholarship prize can go to any of the 30 students, the best leadership qualities prize can then go to any of the *remaining* 29 students, and so on. By the basic principle of counting, the total number of ways to assign prizes is $30 \cdot 29 \cdot 28 \cdot 27 \cdot 26$ (= 17,100,720).

2. A poker hand consists of 5 (necessarily distinct) cards from a deck of 52. So the total number of hands is $\binom{52}{5} = (52 \cdot 51 \cdot 50 \cdot 49 \cdot 48)/(5 \cdot 4 \cdot 3 \cdot 2 \cdot 1) =$ 2,598,960.

3. In going from A to the circled point, one must take exactly four steps, two of which must be rightward and two of which must be upward. Any two of the four steps can be chosen to be the upward steps, so the number of ways to get from A to the circled point must be $\binom{4}{2} = 6$. Likewise, the number of ways to get from the circled point to B must be $\binom{3}{1} = 3$. Each way of getting from A to the circled point, in combination with a way of getting from the circled point to B, gives a different way of getting from A to B via the circled point. Thus, by the basic principle of counting, there are $6 \cdot 3 = 18$ different paths from A to B that go through the circled point.

   Note: Another way to arrive at 6 as the answer to the sub-problem (of counting ways to get from A to the circled point) is as follows: Every such path consists of two rightward steps and two upward steps, interspersed in some order, and can thus be represented by a four-letter "word" consisting of 2 R's and 2 U's (R for Right, U for Up). But this is the same sort of problem as the "Mississippi" example given in the text, so the answer is given by a multinomial coefficient, in this case, $\binom{4}{2,2}$.

4. (a) When there are no constraints on how the teachers are assigned, the first teacher may be assigned to any of the 4 schools, the second teacher may then be assigned to any of the 4 schools, etc. So the number of assignments is $4^8 = 65{,}536$. (b) If each of the four schools must get 2 teachers, then the number of assignments is $\binom{8}{2,2,2,2} = \frac{8!}{2!2!2!2!} = 2520$.

   Note: It is possible to derive the answer to (b) in a different way. First we may choose 2 of the 8 teachers to go to the first school in $\binom{8}{2}$ ways. Then we may choose 2 of the remaining 6 teachers to go to the second school in $\binom{6}{2}$ ways. Then we may choose 2 of the remaining 4 teachers to go to the third school in $\binom{4}{2}$ ways. Lastly we may "choose" 2 of the remaining 2 teachers to go to the fourth school (I put "choose" in quotes since it's no choice at all!). By the basic principle of counting, there are $\binom{8}{2}\binom{6}{2}\binom{4}{2}\binom{2}{2} = 28 \cdot 15 \cdot 6 \cdot 1 = 2520$ ways to make the assignments.

5. (a) If the teams were identified as Team A and Team B (or distinguished in some other fashion), the number of distinguishable assignments would be $\binom{10}{5,5} = 252$. However, in this problem the order of the teams does not matter, so we must divide by a symmetry factor of $2! = 2$, yielding $252/2 = 126$. (b) If the teams were ordered as Team 1 through Team 5 (say), the number of distinguishable assignments would be $\binom{10}{2,2,2,2,2} = 113400$. However, in this problem the order of the teams does not matter, so we must divide by a symmetry factor of $5! = 120$, yielding $113400/120 = 945$.