Math 431, Assignment #3: Solutions

(due 2/15/01 and 2/22/01)

1. If the initial outcome is i and the player wins, then one of the following must occur: the initial outcome is i and the player wins on the 1st roll; the initial outcome is i and the player wins on the 2nd roll; the initial outcome is i and the player wins on the 3rd roll; etc. The events $E_{i,n}$ (with i fixed and n varying) are disjoint, with union $\bigcup_{n=1}^{\infty} E_{i,n} = E_i$; so, by Axiom 3 (page 31), $P(E_i) = \sum_{n=1}^{\infty} P(E_{i,n})$. That is, $P(E_i)$ (the probability that the initial outcome is i and the player wins) is equal to the sum of the probabilities $P(E_{i,n})$ (the probability that the initial outcome is i and the player wins on the nth roll).

   If i is 2, 3, or 12, the player loses instantly, so $P(E_{i,n}) = 0$ for all n. If i is 7, the player wins instantly, so $P(E_{i,1}) = 6/36 = 1/6$ and $P(E_{i,n}) = 0$ for all $n > 1$. If i is 11, the player wins instantly, so $P(E_{i,1}) = 2/36 = 1/18$ and $P(E_{i,n}) = 0$ for all $n > 1$. This leaves the cases $i = 4, 5, 6, 8, 9, 10$. In these cases, we may as well start the infinite sum at $n = 2$ rather than $n = 1$, since the game cannot end before the second roll; that is, $P(E_{i,1}) = 0$ when i is 4, 5, 6, 8, 9, or 10.

   We will consider the first three cases together. There are 36^n different ways for n rolls of the pair of dice to land. In how many of these outcomes do we have an i on the first roll and an i on the nth roll, and with each intervening roll of the pair of dice yielding neither an i nor a 7? The number of ways to roll an i on the first roll is $i-1$; the number of ways to roll an i on the last roll is $i-1$; and on each intervening roll, the number of ways to roll something that is neither an i nor a 7 is $36-(i-1)-6 = 31-i$. So the total number of outcomes that contribute to the event $E_{i,n}$ is $(i-1)^2(31-i)^{n-2}$, and $P(E_{i,n}) = (i-1)^2(31-i)^{n-2}/36^n$. Summing the geometric series, we get $P(E_i) = \sum_{n=2}^{\infty}(i-1)^2(31-i)^{n-2}/36^n = \frac{(i-1)^2}{36^2}/(1-\frac{31-i}{36}) = \frac{(i-1)^2}{36(36-(31-i))} = \frac{(i-1)^2}{36(i+5)}$, which takes on the respective values $\frac{9}{36(9)} = \frac{1}{36}$, $\frac{16}{36(10)} = \frac{2}{45}$, and $\frac{25}{36(11)} = \frac{25}{396}$ for $i = 4$, $i = 5$, and $i = 6$.

(Those of you who have already read ahead to Chapter 3 may have done this a different way, using products of probabilities instead of ratios of cardinalities. That is, instead of computing $P(E_{i,n})$ as $\frac{(i-1)^2(31-i)^{n-2}}{36^n}$, you obtained it as $(\frac{i-1}{36})(\frac{31-i}{36})^{n-2}(\frac{i-1}{36})$.)

The cases 8,9,10 yield the same probabilities as the cases 4,5,6, in reverse order. That is because the number of ways to roll an 8 is the same as the number of ways to roll a 6, etc.

So, $\sum_{i=2}^{12} P(E_i) = 0+0+1/36+2/45+25/396+1/6+25/396+2/45+1/36+1/18+0 = 244/495 = .49292929....$ It's no coincidence that this is so close to one-half; no doubt the rules of craps evolved so as to make the game very close to fair.

2. (a) $(\binom{5}{3}+\binom{6}{3}+\binom{8}{3})/\binom{19}{3} = (10+20+56)/969 = 86/969 = .088751....$ (Alternatively, $\binom{5}{3}$ could be replaced by $\binom{5}{3}\binom{6}{0}\binom{8}{0}$, and so on.)

   (b) $\binom{5}{1}\binom{6}{1}\binom{8}{1}/\binom{19}{3} = 5\cdot 6\cdot 8/969 = 240/969 = 80/323 = .247678....$

   (a′) $(5^3+6^3+8^3)/(19^3) = (125+216+512)/6859 = 853/6859 = .124362...$

   (b′) $(5\cdot 6\cdot 8\cdot 3!)/(19^3) = 1440/6859 = .209943....$ (The 3! comes from the fact that a red ball, blue ball, and green ball can be drawn in 3! different orders: RBG, BRG, etc.)

3. To make the problem one in which all the outcomes in the sample space are equally likely, it helps to imagine that the woman keeps trying keys until she has gone through k of them, regardless of whether or not she's succeeded in opening the door. This applies to both parts (a) and (b).

   (a) The number of outcomes is

$$P_{n,k} = n(n-1)(n-2)\cdots(n-k+2)(n-k+1).$$

   Of these, the favorable outcomes are those in which the first $k-1$ keys are all wrong and the kth key is right; the number of such outcomes is $(n-1)(n-2)\cdots(n-k+1)(1)$. Dividing this by $P_{n,k}$ we get $1/n$. (Do you see a simpler way to get this answer?)

   (b) This time the number of outcomes is n^k, and the number of favorable outcomes is $(n-1)^{k-1}$, so the desired probability is $(\frac{n-1}{n})^{k-1}/n$.

4. With a against b, a has a probability of 5/9 of wininng, since out of the 9 equally likely outcomes, five result in a win for spinner a: (5,3), (5,4), (9,3), (9,4), and (9,8). With b against c, b has a probability of 5/9 of wininng, since out of the 9 equally likely outcomes, five result in a win for spinner b: (4,2), (3,2), (8,2), (8,6), and (8,7). With c against a, c has a probability of 5/9 of wininng, since out of the 9 equally likely outcomes, five result in a win for spinner c: (2,1), (6,1), (6,5), (7,1), and (7,5). So, if you're player B, you can win with probability 5/9 by always choosing the spinner that's clockwise from the one that player A just chose. It's better to be player B (the second player).

   It's amusing to note that the situation changes dramatically if there are three players. Then there are 27 different outcomes, of which 11 lead to victory for the player with disk a, 8 lead to victory for the player with disk b, and 8 lead to victory for the player with disk c. So in the three-player version, it's best to be the *first* player (and to pick disk a)!

5. Part (a), first solution: Let E be the event "at least one die lands on a 6" and let F be the event "the dice land on different numbers". Then $P(F) = \frac{6 \cdot 5}{36} = \frac{30}{36}$ and $P(EF) = \frac{1 \cdot 5 + 5 \cdot 1}{36} = \frac{10}{36}$, so $P(E|F) = P(EF)/P(F) = \frac{10}{36} / \frac{30}{36} = 10/30 = 1/3$.

   Part (a), second solution: Conditional upon the dice landing on two different numbers, the two specific numbers that we see are equally likely to be ANY two of the six possibilities. So when we condition upon seeing two different numbers on the two rolls, we see that the two numbers that we get (viewed as an unordered pair) are governed by the uniform distribution on the set of all $\binom{6}{2}$ pairs of numbers between 1 and 6. So the question is equivalent to: If we choose two numbers (without replacement) from $1, \ldots, 6$, what's the chance that one of them is a 6? In this new problem, the sample space has size $\binom{6}{2} = 15$ and the set of "unfavorable" outcomes (those in which neither of the two numbers is a 6) has size $\binom{5}{2} = 10$; hence the probability of an unfavorable outcome is $10/15 = 2/3$ and the probability of a favorable outcome is $1 - 2/3 = 1/3$.

   Part (b), first solution: Use the same notation as above, so that E is the event "at least one die lands on a 6" and F^c is the event "the dice land on the same numbers". Then $P(F^c) = 1 - \frac{5}{6} = \frac{1}{6}$ and $P(EF^c) = \frac{1}{36}$, so $P(E|F^c) = P(EF^c)/P(F) = \frac{1}{36}/\frac{1}{6} = 6/36 = 1/6$.

Part (b), second solution: Conditional upon the dice landing on the same number twice, the specific number that we see is equally likely to be ANY of the six possibilities. So the probability that the specific number that we see twice is a 6 must be 1/6.

6. Part (a), first solution: The probability that the first ball is white is $\frac{6}{15}$ (since 6 of the 15 balls in the urn are white). The probability that the second ball is white, given that the first was white, is $\frac{5}{14}$ (since 5 of the 14 balls remaining in the urn are white). The probability that the third ball is black, given that the first two were white, is $\frac{9}{13}$ (since 9 of the 13 balls remaining in the urn are black). The probability that the fourth ball is black, given that the first two were white and the third was black, is $\frac{8}{12}$ (since 8 of the 12 balls remaining in the urn are black). So, by the multiplication rule, the probability that the first two balls are white and the next two are black is $\frac{6}{15}\frac{5}{14}\frac{9}{13}\frac{8}{12} = \frac{6}{91}$.

   Part (a), second solution: Instead of making an ordered selection of four balls without replacement, one could make an unordered selection of four balls and then order them. (This would be a different setup, but the outcomes would all have the same probabilities under this setup as they would under the original setup, so we can use this alternative setup for doing our calculations.) The probability that two of the four balls that are chosen are white and the other two are black is $\binom{6}{2}\binom{9}{2}/\binom{15}{4} = 15 \cdot 36/1365 = 36/91$. If one orders those four balls randomly (assuming two are white and two are black), there are 6 equally likely orderings, only 1 of which is white-white-black-black. So the conditional probability of drawing the balls in the order white-white-black-black, given that two are white and two are black, is 1/6. Hence, by the multiplication rule, the probability that the first two balls are white and the next two balls are black is $\frac{36}{91}\frac{1}{6} = \frac{6}{91}$.

   Part (b), first solution: The probability that the first ball is black is $\frac{9}{15}$ (since 9 of the 15 balls in the urn are black). The probability that the second ball is white, given that the first was black, is $\frac{6}{14}$ (since 6 of the 14 balls remaining in the urn are white). The probability that the third ball is black, given that the first was black and the second was white, is $\frac{8}{13}$ (since 8 of the 13 balls remaining in the urn are black). The probability that the fourth ball is white, given that the first ball was black, the second was white, and the third was black, is $\frac{5}{12}$ (since 5 of the 12 balls remaining in the urn are black). So, by the multiplication rule, the probability that the first two balls are white and the next two are black is $\frac{9}{15}\frac{6}{14}\frac{8}{13}\frac{5}{12} = \frac{6}{91}$, as before.

Part (b), second solution: Let's make an unordered selection of four balls and then order them, as in the alternative solution to part (a). As in (a), the probability of drawing two balls of each color is 36/91. There are 6 equally likely orderings of the four balls, only 1 of which is black-white-black-white. So the conditional probability of drawing the balls in the order black-white-black-white, given that two are white and two are black, is 1/6. Hence, by the multiplication rule, the probability that the first two balls are white and the next two balls are black is $\frac{36}{91}\frac{1}{6} = \frac{6}{91}$.

7. Let E be the event of an ectopic pregnancy, and S be the event that the mother smokes. We are given that $P(E|S) = 2P(E|S^c)$ and $P(S) = 0.32$. Write $P(E|S^c) = p$ and $P(E|S) = 2p$. By Bayes' Theorem, $P(S|E) = P(S)P(E|S)/[P(S)P(E|S) + P(S^c)P(E|S^c)] = 2pP(S)/[2pP(S) + pP(S^c)] = 2P(S)/[2P(S) + P(S^c)]$ (note that the p's cancel!), which equals $0.64/[0.64 + 0.68] = 0.4848....$

8. Initially, there are 4 aces and 48 non-aces in the deck, so $P(E_1) = \binom{4}{1}\binom{48}{12}/\binom{52}{13} = 9139/20825$. If the first hand has 1 ace, then there are 3 aces and 36 non-aces remaining in the deck, so $P(E_2|E_1) = \binom{3}{1}\binom{36}{12}/\binom{39}{13} = 325/703$. If the first and second hands have 1 ace apiece, then there are 2 aces and 24 non-aces remaining in the deck, so $P(E_3|E_1E_2) = \binom{2}{1}\binom{24}{12}/\binom{26}{13} = 13/25$. Finally, if the first three hands have 1 ace apiece, then there are 1 ace and 12 non-aces remaining in the deck, so $P(E_4|E_1E_2E_3) = \binom{1}{1}\binom{12}{12}/\binom{13}{13} = 1$ (which in fact is obvious: if the first three players each got 1 of the 3 aces, the fourth player is forced to get exactly 1 ace). Putting all this together, we get $P(E_1E_2E_3E_4) = P(E_1)P(E_2|E_1)P(E_3|E_1E_2)P(E_4|E_1E_2E_3) = \frac{9139}{20825}\frac{325}{703}\frac{13}{25} = \frac{2197}{20825}$, which is approximately 0.105498.

9. If $P(A)/(1 - P(A)) = \alpha$, then $P(A) = (1 - P(A))\alpha = \alpha - P(A)\alpha$,

so $P(A) + P(A)\alpha = \alpha$. Writing this as $P(A)(1 + \alpha) = \alpha$, we get $P(A) = \frac{\alpha}{1+\alpha}$. (Check: When the odds ratio is $1 : 1$, we get $\alpha = 1$; the formula gives $P(A) = \frac{1}{1+1} = \frac{1}{2}$, which is correct.)