Math 431, Assignment #5: Solutions

(due 3/22/01)

1. We already know that X, Y, and Z each have expected value 3, so $\text{Var}(X) = (1-3)^2(\frac{1}{3}) + (4-3)^2(\frac{2}{3}) = 2$, $\text{Var}(Y) = \text{Var}(X) = 2$, and $\text{Var}(Z) = (2-3)^2(\frac{2}{3}) + (5-3)^2(\frac{1}{3}) = 2$. We already know that $X+Y$ and $X+Z$ each have expected value 6, so $\text{Var}(X+Y) = (2-6)^2(\frac{1}{9})+(5-6)^2(\frac{4}{9})+(8-6)^2(\frac{4}{9}) = 4$ and $\text{Var}(X+Z) = (3-6)^2(\frac{2}{9})+(6-6)^2(\frac{5}{9})+(9-6)^2(\frac{2}{9}) = 4$. (Note that these numbers agree with the theorem that tells us that $\text{Var}(X+Y) = \text{Var}(X)+\text{Var}(Y)$ when X and Y are independent and $\text{Var}(X+Z) = \text{Var}(X) + \text{Var}(Z)$ when X and Z are independent.) Taking the square roots of all these numbers, we get $\sigma(X) = \sigma(Y) = \sigma(Z) = \sqrt{2}$ and $\sigma(X+Y) = \sigma(X+Z) = \sqrt{4} = 2$.

2. (a) Let Y denote the outcome of a single spin. With spinner A, we get $E(Y) = (1)(0.5) + (4)(0.5) = 2.5$ and $E(Y^2) = (1)(0.5) + (16)(0.5) = 8.5$, so $\text{Var}(Y) = E(Y^2) - [E(Y)]^2 = 8.5 - 6.25 = 2.25$. So, if we let Y_i denote the outcome of the ith spin, we get $E(Y_i) = 2.5$ and $\text{Var}(Y_i) = 2.25$ for $i = 1$ to 5. We have $X = Y_1 + Y_2 + Y_3 + Y_4 + Y_5$. By linearity of expectation, $E(X) = E(Y_1) + E(Y_2) + E(Y_3) + E(Y_4) + E(Y_5) = 5(2.5) = 12.5$, and since the Y_i's are independent, $\text{Var}(X) = \text{Var}(Y_1) + \text{Var}(Y_2) + \text{Var}(Y_3) + \text{Var}(Y_4) + \text{Var}(Y_5) = 5(2.25) = 11.25$.

   With spinner B, we get $E(Y) = (2)(0.5) + (3)(0.5) = 2.5$ and $E(Y^2) = (4)(0.5) + (9)(0.5) = 6.5$, so $\text{Var}(Y) = E(Y^2) - [E(Y)]^2 = 6.5 - 6.25 = 0.25$, so $E(X) = 5(2.5) = 12.5$ and $\text{Var}(X) = 5(0.25) = 1.25$.

   Note that the expected value of X is 12.5 for both spinners, but that the variance of X is significantly greater for spinner A.

   (b) Say that "success" occurs on a given spin if I spin the larger of the two numbers shown on the spinner I am using (an event that occurs with probability 1/2, regardless of which spinner I use). With spinner A, the sum of the five spins will be 14 or larger provided I get three or more successes. With spinner B, the sum will be 14 or larger provided I get four or more successes. Since the probability of getting three or more successes is greater than the probability of getting four or more successes, I am better off with spinner A. The respective probabilities are given by the binomial distribution: the probability of three or more successes is $\binom{5}{3}(\frac{1}{2})^5 + \binom{5}{4}(\frac{1}{2})^5 + \binom{5}{5}(\frac{1}{2})^5 = (10+5+1)/32 = 1/2$ and the probability of four or more successes is $\binom{5}{4}(\frac{1}{2})^5 + \binom{5}{5}(\frac{1}{2})^5 = (5+1)/32 = 3/16$.

(c) With spinner A, the sum will be 11 or larger provided I get two or more successes. With spinner B, the sum will be 11 or larger provided I get one or more successes. Since the probability of getting two or more successes is less than the probability of getting one or more successes, I am better off with spinner B. We can compute the respective probabilities as above, but the calculation is slightly simpler if we compute the complementary probabilities. That is: the probability of one or fewer successes is $\binom{5}{1}(\frac{1}{2})^5 + \binom{5}{0}(\frac{1}{2})^5 = (5+1)/32 = 3/16$, so the probability of two or more successes is $1-3/16 = 13/16$; and the probability of "zero or fewer" successes is $1/32$, so the probability of one or more successes is $1 - 1/32 = 31/32$.

What to notice: For both spinners, the expected value of X is 12.5. If we're hoping for X to lie in some range (14 and above) that doesn't include the expected value, then we are essentially counting on a fluke, and we want the variance to be large (so we pick spinner A). On the other hand, if we're hoping for X to lie in some range (11 and above) that *does* include the expected value, then we are essentially counting on the law of averages, and we want the variance to be small (so we pick spinner B).

3. Fix k between 1 and 10. The number of floors at which the elevator stops can be written as $X_1 + \ldots + X_{10}$, where X_k is the indicator function for the event that at least one person chooses floor k. Hence the expected number of floors is $E(X_1) + \ldots + E(X_{10})$. Note that $E(X_k)$ is just the probability that the elevator will stop on the kth floor, which is $1 - (.9)^{12}$ (since $(.9)^{12}$ is the probability that all twelve of the passengers choose a floor different from k). Letting k run from 1 to 10 and summing, we see that the expected number of stops is $10(1 - (.9)^{12})$, which is approximately 7.2.

   Alternatively: The probability that the elevator will not stop on the

kth floor is $(.9)^{12}$, so the expected number of floors that the elevator will not stop at is 10 times $(.9)^{12}$, and the expected number of floors that the elevator *will* stop at is 10 minus that, or $10 - 10(.9)^{12}$.

4. (a) $E(X) = \sum_{n=1}^{\infty}(\alpha^n)(\frac{1}{2})^n = \sum_{n=1}^{\infty}(\alpha/2)^n$; this geometric series converges as long as $\alpha/2 < 1$, i.e., $\alpha < 2$.

   (b) $E(X^2) = \sum_{n=1}^{\infty}(\alpha^{2n})(\frac{1}{2})^n = \sum_{n=1}^{\infty}(\alpha^2/2)^n$; this geometric series converges as long as $\alpha^2/2 < 1$, i.e., $\alpha < \sqrt{2}$.

   (c) If α is 1.5 (which lies between 2 and $\sqrt{2}$), then by part (a), $E(X)$ is finite. On the other hand, $E(X^2)$ will be infinite, so $\mathrm{Var}(X) = E(X^2) - [E(X)]^2$ will be infinite too.

5. Suppose the newsboy currently buys k papers each day, and is considering buying $k+1$ papers each day instead. Is this a smart move? Each day, he will have to expend an additional 10 cents, but he stands to make an extra profit of 15 cents in the event that the demand for papers that day is greater than k; that is, his expected increase in revenue is $15P(X > k)$, where X is the demand. Buying the $k =$ 1st paper will be a smart move provided $15P(X > k) > 10$, that is, provided $P(X > k) > 2/3$, or equivalently, $P(X \leq k) \leq 1/3$. Since X is a binomial random variable with parameters $n = 10$ and $p = 1/3$, we have

$$\begin{aligned}
P(X=0) &= \binom{10}{0}(1/3)^0(2/3)^{10} = .0173, \\
P(X=1) &= \binom{10}{1}(1/3)^1(2/3)^9 = .0867, \\
P(X=2) &= \binom{10}{2}(1/3)^2(2/3)^8 = .1951, \\
P(X=3) &= \binom{10}{3}(1/3)^3(2/3)^7 = .2601, \\
&\dots
\end{aligned}$$

Taking sums, we have

$$\begin{aligned}
P(X \leq 0) &= .0173, \\
P(X \leq 1) &= .0173 + .0867 = .104,
\end{aligned}$$

$$
\begin{aligned}
P(X \leq 2) &= .0173 + .0867 + .1951 = .299, \\
P(X \leq 3) &= .0173 + .0867 + .1951 + .2601 = .559, \\
&\dots
\end{aligned}
$$

Note that when k is less than 3, $P(X \leq k) < 1/3$, so buying an additional paper is a good idea; but for all k greater than or equal to 3, $P(X \leq k) > 1/3$, so buying an additional paper just makes the newsboy's expected profit lower. Hence buying 3 papers each day is his best bet, in the long run.

Note that when I add two numbers known to within an error of .0001, the sums are not necessarily correct to within an error of .0001. However, as long as I add no more than 10 such numbers, the sum will be correct to within an error of .001. That's why the partial sums in the second table are given to only three significant figures.

6. If we assume that the occurrence of a transmission error on one bit is independent of occurrence of transmission errors on any of the other bits, then the binomial distribution applies, and the probability of exactly k bits being incorrectly received is $\binom{5}{k}(.2)^k(.8)^{5-k}$. The message will be decoded incorrectly if the number of incorrectly received bits is 3, 4, or 5. Therefore the probability that the message will be decoded incorrectly is $\binom{5}{3}(.2)^3(.8)^2 + \binom{5}{4}(.2)^4(.8)^1 + \binom{5}{5}(.2)^5(.8)^0$ or $.05120 + .00640 + .00032 = .05792$ (roughly a 6 percent chance).

7. The probability that the stronger team wins in exactly i games is equal to $\binom{i-1}{3}$ times $(.6)^4(.4)^{i-4}$; that is, the number of ways in which the team could win 3 of the first $i-1$ games and then have its 4th win in the next game, times the probability of the team winning any 4 particular games and losing the other $i-4$ of the first i games. Evaluating this explicitly for $i = 4$ through $i = 7$, we get .1296, .20736, .20736, and .165888, which sum to .7102 (to four significant figures).

   On the other hand, if the two teams played a best-of-three match, the probability of the stronger team winning would be $\binom{1}{1}(.6)^2 + \binom{2}{1}(.6)^2(.4) = .36 + .288 = .648$. So, the probability of the stronger team winning would decrease from 71 percent to 65 percent. (This makes intuitive sense: the more games are played, the more likely it is that the stronger team will prevail. Note also that if we switched to a "best-of-one" match, i.e., a single deciding game, the probability of the stronger team winning goes all the way down to .6.)

Now we switch to the case where the two teams (call them A and B) are evenly matched. The probability that team A wins in 4 games is $(1/2)^4$, as is the probability that team B wins in 4 games. Since these two events are mutually exclusive, the probability that the match ends in 4 games is $2(1/2)^4 = 1/8$. Likewise, the probability that the match ends in 5 games is $2\binom{4}{3}(1/2)^4(1/2) = 1/4$, the probability that the match ends in 6 games is $2\binom{5}{3}(1/2)^4(1/2)^2 = 5/16$, and the probability that the match ends in 7 games is $2\binom{6}{3}(1/2)^4(1/2)^3 = 5/16$. So the expected number of games played is $(4)(1/8)+(5)(1/4)+(6)(5/16)+(7)(5/16) = 93/16 = 5.8125$.

8. (a) Here the probability that the interviewer will find enough interviewees is just the probability that all 5 people will say "yes", which is $(2/3)^5 = 32/243 = .1317$.

   (b) We can imagine that the interviewer keeps asking people if they're willing to be interviewed, even after she's found five who say "yes". Then the question is, did she have 5 or more successes in 8 trials? This is given by a binomial distribution, and the answer is $\binom{8}{5}(2/3)^5(1/3)^3 + \binom{8}{6}(2/3)^6(1/3)^2 + \binom{8}{7}(2/3)^7(1/3) + \binom{8}{8}(2/3)^8 = 4864/6561 = .7414$. Alternatively, we can imagine that the interviewer stops after she's found five people who say "yes". Then the question is, did 8 trials suffice to give 5 successes? This is given by the negative binomial distribution, and the answer is $\binom{4}{4}(2/3)^5 + \binom{5}{4}(2/3)^5(1/3) + \binom{6}{4}(2/3)^5(1/3)^2 + \binom{7}{4}(2/3)^5(1/3)^3 = 4864/6561 = .7414$.

   (c) $\binom{5}{4}(2/3)^5(1/3) = .2195$.

   (d) $\binom{6}{4}(2/3)^5(1/3)^2 = .2195$.

9. $E[Y] = E[X/\sigma-\mu/\sigma] = E[X/\sigma]-\mu/\sigma = E[X]/\sigma-\mu/\sigma = \mu/\sigma-\mu/\sigma = 0$, and $\mathrm{Var}[Y] = \mathrm{Var}[X/\sigma - \mu/\sigma] = \mathrm{Var}[X/\sigma] = \frac{1}{\sigma^2}\mathrm{Var}[X] = \frac{1}{\sigma^2}\sigma^2 = 1$.

10. When differentiating a product (or a ratio of products), it's easiest to first take the log before differentiating. In this case, we want to find

the p that maximizes $\log P(X=k)$. $\log P(X=k) = \log \binom{n}{k} + k \log p + (n-k)\log(1-p)$ so, $\frac{d}{dp} \log P(X=k) = \frac{k}{p} - \frac{n-k}{1-p}$. Setting this equal to 0, we find $p = k/n$ is the only critical point for $\log P(X=k)$ (and hence for $P(X=k)$ as well). In fact, $p = k/n$ is a local maximum, since the value of $P(X=k)$ for $p = k/n$ is positive whereas the value of $P(X=k)$ for the endpoints $p=0$ and $p=1$ both vanish. (Note that this last claim assumes that k lies strictly between 0 and n; but the extreme cases $k=0$ and $k=n$ are trivial, since in these cases when we put $p=k/n$ we can get $P(X=k)$ to equal 1, which is clearly the best we could hope for.)