Math 431, Quiz #1: Solutions

1. 25 points A certain probability class contains 32 students, among whom are two Chads, two Davids, two Kevins, two Pauls, and two Peters (along with 22 students whose names are all different from these five names and from one another's names).

(a) How many different 16-student study-groups can be formed such that within the study group, there is exactly one Chad, exactly one David, exactly one Kevin, exactly one Paul, and exactly one Peter?

(b) If the teacher randomly assigns 16 of the students to a study-group, what is the chance that this study group will contain both Chads, both Davids, both Kevins, both Pauls, and both Peters?

Solution:
(a) To create such a study group, one must pick 1 of the 2 Chads, 1 of the 2 Davids, 1 of the 2 Kevins, 1 of the 2 Pauls, 1 of the 2 Peters, and 11 of the other 22 students. Therefore the number of such study-groups is $\binom{2}{1}\binom{2}{1}\binom{2}{1}\binom{2}{1}\binom{2}{1}\binom{22}{11}$.
(b) Likewise, the number of study-groups with both Chads, both Davids, etc., is $\binom{2}{2}\binom{2}{2}\binom{2}{2}\binom{2}{2}\binom{2}{2}\binom{22}{6} = \binom{22}{6}$. The total number of possible 16-student study-groups is $\binom{32}{16}$. Hence the probability that a random 16-student study-group will contain both Chads, both Davids, etc. is $\binom{22}{6}/\binom{32}{16}$.
Remarks:

- Many students confused "How many...?" (part (a)) with "What is the chance...?" (part (b)). The answer to the first is a counting number, that is, a non-negative integer; the answer to the second is a probability, that is, a real number between 0 and 1.

- For part (a), some students divided by 2, because the complement of a study-group with 1 Chad, 1 David, etc. is also a study-group with 1 Chad, 1 David, etc. However, it is a different study-group, and should be counted as different, so it is not appropriate to divide by 2. (If the question had been "In how many ways is it possible to divide the class into two study-groups, each of which has 1 Chad, 1 David, etc.?", then it would have been appropriate to divide by 2.)

- Two common wrong answers to (b) were $\binom{22}{16}/\binom{32}{16}$ and $\binom{22}{11}/\binom{32}{16}$. The former ignores the fact that only 6 of the original 16 slots remain vacant after the Chads, Davis, et al. have been assigned to the study-group; the latter tries to take this fact into account but miscounts the number of vacant slots.

- Some students tried to answer part (b) by using a sample space in which order counts (a sample space of size $P_{32,16} = 32 \cdot 31 \cdots 17$). One can do the problem this way (though it is much more laborious than the way given above), if one is consistent in one's assumptions. However, if the sample space has been chosen to consist of all ways of ordering 16 students from the class of 32, it is not consistent with our choice of sample space to assume that in the favorable outcomes, the first two students chosen are Chads, the next two are Davids, etc.

2. 25 points

(a) I pick three cards from a deck, without replacement. (Recall that in a deck of 52 cards, there are 13 cards of each suit.) In how many different ways can it happen that two are of the same suit and one is of a different suit, if order matters?

(b) In how many ways can I get two cards of the same suit and one of a different suit, if order *doesn't* matter (i.e., if outcomes that involve the same three cards drawn in a different order are regarded as being the same outcome)?

Solution: Say that the two cards of the same suit are of the "long suit" and that the remaining card is of the "short suit".
(a) To specify a three-card hand of the specified kind, one can proceed as follows. First specify the long suit (4 possibilities); then specify the short suit (3 possibilities); then specify the two cards in the long suit ($\binom{13}{2} = 78$ possibilities); then specify the card in the short suit ($\binom{13}{1} = 13$ possibilities); then choose how to order the 3 cards ($3! = 6$ possibilities). Hence the total number of such hands, if order matters, is $4 \times 3 \times 78 \times 13 \times 6 = 73008$.
(b) One can choose any of the 4 suits to be the long suit. Then one can choose two cards in the long suit in $\binom{13}{2} = 78$ ways, and one card not in the

long suit in 39 ways. Hence the total number of such hands, if order does not matter, is $4 \times 78 \times 39 = 12168$.

Remarks:

- Each unordered hand consisting of two cards in one suit and one card in another suit corresponds to 6 different ordered hands satisfying the same constraint. Hence the answer to part (a) can be calculated as the answer to part (b), times 6; or, the answer to part (b) can be calculated as the answer to part (a), divided by 6.

- Here is another solution to part (a): There are 52 possibilities for the first card. If the second card is of the same suit (which can happen in 12 ways), the third card must be of a different suit (which can happen in 39 ways); if the second card is not of the same suit as the first (which can happen in 39 ways), the third card must be of the same suit as either the first card or the second (which can happen in $12 + 12 = 24$ ways). Hence the total number of such hands, if order matters, is $52(12 \times 39 + 39 \times 24) = 73008$.

- Most students who got the wrong answer used some variant of the preceding arguments. and left out some relevant factor from the product. For example, the answer $3!\binom{13}{2}\binom{13}{1}$ (for part (a)) arose for those students who forgot that they have to specify which suit is long and which is short.

- Some students gave $52 \times 12 \times 39 \times 3!$ as the answer to (a), probably reasoning as follows: "There are 52 possibilities for the first card, 12 possibilities for the second card to be of the same suit, and 39 possibilities for the third card to be of a different suit. But $52 \times 12 \times 39$ is not the right answer yet, because it assumes the cards came in a particular order. To get the correct answer, we multiply by the symmetry factor $3! = 6$." Note, however, that the provisional answer $52 \times 12 \times 39$ assumes only that the card in the short suit was the third card picked; it assumes nothing about the first two cards. Since the card in the short suit could be the first, second, or third card drawn, the appropriate symmetry factor is 3, not 3!.

- A few students, in answering part (b), assumed that the two cards from the long suit were drawn first.

3. 25 points I roll a pair of (fair) dice 3 times. What is the probability that, on at least one of the three tries, I roll a 7?

Solution: The probability that I fail to roll a 7, on any particular try, is 5/6. Hence the probability that I fail on all three tries is $(5/6)^3 = 125/216$, and the probability that I succeed on at least one try is $1 - 125/216 = 91/216$.

Remarks:

- One wrong answer given was $1/6+1/6+1/6$. This incorrectly assumes that the three events (7 on the first roll, 7 on the second roll, 7 on the third roll) are disjoint (because the probability of a union of three events is equal to the sum of the probabilities of the three individual events only if the three events are disjoint, i.e., no two of them intersect). One way to see that this reasoning is incorrect is to notice that if we replaced "three rolls" by "seven rolls", addition of probabilities would give $1/6 + 1/6 + 1/6 + 1/6 + 1/6 + 1/6 + 1/6 = 7/6$, which cannot be a probability since it is greater than 1.

- $1/6 + 1/6 + 1/6$ is the right answer to a different question, namely, "What is the expected number of 7's that will be rolled in three tries?"

- $1/6+1/6+1/6$ can also be viewed as the first step in a correct solution to the original problem, by way of the principle of inclusion-exclusion. This principle says that $P(E \cup F \cup G) = P(E)+P(F)+P(G)-P(EF)-P(EG)-P(FG)+P(EFG)$. In this case, we get $1/6 + 1/6 + 1/6 - 1/36 - 1/36 - 1/36 + 1/216 = 91/216$.

- Another way to get the right answer is $3(\frac{1}{6} \cdot \frac{5}{6} \cdot \frac{5}{6}) + 3(\frac{1}{6} \cdot \frac{1}{6} \cdot \frac{5}{6}) + (\frac{1}{6} \cdot \frac{1}{6} \cdot \frac{1}{6})$, where the terms respectively give the probability of exactly one 7, the probability of exactly two 7's, and the probability of exactly three 7's. The first coefficient of 3 takes into account the fact that there are 3 different positions in which the 7 could occur, and the second coefficient of 3 takes into account the fact that there are 3 different positions in which the non-7 could occur.

- $(1/6)^3$ is the right answer to a different question, namely, what is the probability that I roll a 7 on all three tries.

4. 25 points Suppose that one child out of forty-three suffers from a particular disease that only emerges when the child reaches adulthood. There is a (not completely reliable) test for the condition: when a child who has the disease takes the test, it comes up positive 80% of the time, and when a child who does not have the disease takes the test, it comes up positive only 10% of the time. What is the chance that an individual child has the disease, given that the child tested positive? Give your answer as both a probability and an odds ratio.

Solution: Let D be the event that a child has the disease, and T be the event that a child tests positive for the disease. By Bayes' Formula, $P(D|T) = \frac{P(D)P(T|D)}{P(D)P(T|D)+P(D^c)P(T|D^c)} = \frac{(1/43)(0.8)}{(1/43)(0.8)+(42/43)(0.1)}$. Multiplying top and bottom by 43, we get $\frac{(1)(0.8)}{(1)(0.8)+(42)(0.1)} = \frac{0.8}{5.0} = 0.16$. So a child who tests positive for the disease has a 16 percent chance of actually having it; that is, the probability is 0.16. The odds ratio is 16 percent divided by $100 - 16 = 84$ percent, or $16/84 = 0.19$.

Remarks:

- The answer, to six significant figures, is 0.190476, so 0.19, 0.190, 0.1905, 0.19048, etc., are all admissible answers; however, 0.1903, 0.1904, 0.19047, etc., are not. When one gives a quantity to four decimal places, without a proviso about the magnitude of possible error, one is asserting that one's estimate is correct to four decimal places. A spurious precision of precision is misleading and should be avoided.

- An odds ratio is a ratio, not a probability. (Note that the odds ratio can be bigger than 1; a probability never can.) Hence, it is wrong to stay that the odds ratio is "a 19 percent chance". It is, however, correct to say that the odds ratio is 19 percent.

- One could also describe the odds ratio as 16-to-84 odds, or 4-to-21 odds.