Math 567, Assignment #1
(due February 3)

NOTE: Please record on your paper the amount of time you spent working on and writing up the homework, so that I can keep control on the sizes of the assignments. Feel free to work with others, and to see me for hints when all else fails.

As in all assignments, you are expected to show your work. When words like "prove" or "show" are used in the statement of a problem, you should use roughly the same amount of detail as are used in proofs in the textbook.

1. For all $n \geq 1$, prove the following by mathematical induction:

$$\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + ... + \frac{1}{n^2} \leq 2 - \frac{1}{n} \ .$$

2. For each of the following four Diophantine equations, determine all solutions in the integers, and list all the solutions in the positive integers: (a) $18x+5y = 48$; (b) $54x+21y = 906$; (c) $123x+360y = 99$; (d) $158x-57y = 7$. (If there are infinitely many solutions in positive integers, list them in size order, starting from the smallest, and make the governing pattern clear.)

3. For any integer a, show the following: (a) $\gcd(2a+1, 9a+4) = 1$; (b) $\gcd(5a+2, 7a+3) = 1$; (c) If a is odd, then $\gcd(3a, 3a+2) = 1$.

4. Find (with proof) all values of n such that $\frac{n^2+1}{n+2}$ is an integer.

5. (a) Show that a number of the form $2^n + 1$ (with $n > 1$) can be prime only if n is a power of 2. (Hint: For m odd, we know how to write a sum of two mth powers in factored form.) (b) Show that all numbers of the form $2^{(2^k)} + 1$ are mutually relatively prime.

6. Show that a number of the form $2^n - 1$ (with $n > 1$) can be prime only if n is prime. (Hint: Show that $2^m - 1$ divides $2^n - 1$ when m divides n.)

7. Show that there are infinitely many primes in the infinite arithmetic progression 2, 5, 8, 11, 14, 17, ... (Hint: You can model your proof on Burton's proof of Theorem 3.6, which asserts that there are infinitely primes of the form $4k + 3$.)