Math 567, Assignment #12

(due Thursday, May 11)

1. Show that if $n \equiv 3$ or $6 \pmod 9$, then n cannot be represented as a sum of two squares.

2. The Corollary to Theorem 12.3 asserts that a positive integer n is representable as the sum of two squares if and only if each of its prime factors of the form $4k+3$ occurs with even exponent. Making use of this fact if necessary, show that if the positive integer n cannot be written as the sum of the squares of two integers, then n cannot be written as the sum of the squares of two rational numbers.

3. Suppose that s and t are positive integers satisfying $s > t > 0$ and $\gcd(s,t) = 1$, with s and t *both odd* (compare this with the situation in Theorem 11.1, where exactly one of s, t is odd). Let $x = 2st$, $y = s^2 - t^2$, and $z = s^2 + t^2$. Show that $\frac{x}{2}, \frac{y}{2}, \frac{z}{2}$ is a primitive Pythagorean triple.

4. Show, in two different ways, that if an odd prime number p can be written in the form $p = a^2 + 2b^2$ for some choice of integers a, b, then p must be congruent to 1 or 3 (mod 8). To be specific, first prove the claim by considering the equation modulo 8; then re-prove the claim by using the laws of quadratic reciprocity (as we did in class for primes of the form $p = a^2 + b^2$).

5. Show that if the prime number p is congruent to 1 or 3 (mod 8), then p can be written in the form $p = a^2 + 2b^2$ for some choice of integers a, b. (Hint: Mimic the proof of Theorem 12.2 given in lecture.)

6. Call a solution of the equation $x^2 + 2y^2 = z^2$ "primitive" if $\gcd(x,y,z) = 1$. Show that a triple of positive integers x, y, z is a primitive solution of this equation *if and only if* there exist relatively prime positive integers s, t with s odd, satisfying $x = |s^2 - 2t^2|$, $y = 2st$, and $z = s^2 + 2t^2$.