Math 567, Assignment #11: Solutions

1. The equation $x(x+1)/2 = 2 \cdot y(y+1)/2$ can be rewritten as $4x^2 + 4x = 2(4y^2+4y)$ or $(4x^2+4x+1) = 2(4y^2+4y+1)-1$ or $a^2 = 2b^2-1$ where $a = 2x+1$ and $b = 2y+1$. Conversely, our algebra shows that if a, b are odd integers satisfying $a^2 = 2b^2-1$, then by putting $x = (a-1)/2$ and $y = (b-1)/2$ we can obtain a pair of integers x, y satisfying $x(x+1)/2 = 2y(y+1)/2$. So, from our list of whole-number solutions to the equation $a^2 - 2b^2 = \pm 1$ (demonstrated during lecture), we need to extract the ones in which $a^2 - 2b^2 = -1$ and both a and b are odd. By considering the equation $a^2 - 2b^2 = -1$ modulo 4, we see that the stated conditions on a, b are redundant: the only way to have $a^2 - 2b^2 = -1$ is if a and b are both odd. So we need merely require that $a^2 - 2b^2 = -1$. But we already saw in lecture that the sign of $a^2 - 2b^2$ alternates, so we simply need to take every other convergent: 1/1, 7/5, 41/29, 239/169, etc. These gives rise to the solutions (x, y) = (0,0) (boring), (3,2) (old), (20,14), (119,84), and so on ad infinitum. (The numbers clearly get larger and larger, so the sequence never repeats.) Hence the pairs of triangle numbers we seek are: 6 and 3; 210 and 105; 7140 and 3570; etc.

2. The positive root of the equation $x^2 - 3x - 1 = 0$ is $(3 + \sqrt{13})/2$; its continued fraction $3 + 1/3 + 1/3 + ...$ has convergents 3/1, 10/3, 33/10, etc. Writing these as x/y, we respectively obtain $x^2 - 3xy - y^2$ equal to -1, $+1$, -1, and it is easy to show by induction that they alternate in sign forever: the base case is trivial, and to check the induction step, we use the recurrences $x' = 3x+y$ and $y' = x$ to get $x'^2 - 3x'y' - y'^2 = (3x+y)^2 - 3(3x+y)(x) - x^2 = -x^2 + 3xy + y^2 = -(x^2 - 3xy - y^2)$. (Note that this calculation makes sense, and demonstrably gives us a way of generating new solutions from old, even if we don't bother to show that the continued fraction is indeed as claimed.) So three solutions are (a, b) = (10,3), (109,33), and (1189,360). (Here are taking only half of the convergents: the others, such as (33,10), satisfy $x^2 - 3xy - y^2 = -1$ instead of $x^2 - 3xy - y^2 = +1$ and hence are not what we seek.)

If we want to verify the claimed continued fraction expansion, we can do it as follows: if we put $x = 3 + 1/3 + 1/3 + ...$ (which, by the remarks leading up to definition 14.3, has some definite positive value), we get $x = 3 + 1/x$, which implies $x^2 - 3x - 1 = 0$, whose only positive root is $(3 + \sqrt{13})/2$; this verifies the claimed continued fraction expansion.