Math 567, Assignment #3: Solutions

1. (a) One way is to solve by a kind of nested trial-and-error procedure. If $x \equiv 1 \pmod 3$, then x must be congruent to 1, 4, 7, 10, or 13 (mod 15). Of these, only 7 is congruent to 2 (mod 5). So, any solution to this problem must be congruent to 7, 22, 37, 52, 67, 82, or 97 (mod 105). Of these, only 52 is congruent to 3 (mod 7). So our general solution is $x \equiv 52 \pmod{105}$ (that is, $x = 52 + 105k$ for some k).

(b) Another way to solve is by noticing patterns and being clever. Doubling all the congruences given in part (b), we get $2x \equiv 10 \pmod{11}$, $2x \equiv 28 \pmod{29}$, and $2x \equiv 30 \pmod{31}$. That is, $2x$ is 1 less than a multiple of 11, 1 less than a multiple of 29, and 1 less than a multiple of 31. Hence we may take $2x$ to be 1 less than $11 \times 29 \times 31$, that is, $2x = 9888$, whence our particular solution is $x = 4944$ and the general solution is $x \equiv 4944 \pmod{9889}$.

(c) When all else fails (and especially when the numbers involved are large), there's always the method of superposition. To obtain our three fundamental solutions for the general problem of solving simultaneous linear congruences modulo 6, 11, and 17, we need to solve the congruences $(11)(17)x \equiv 1 \pmod 6$, $(6)(17)y \equiv 1 \pmod{11}$, and $(6)(11)z \equiv 1 \pmod{17}$. Since $11 \times 17 \equiv 1 \pmod 6$ and $6 \times 17 \equiv 3 \pmod{11}$ and $6 \times 11 \equiv 15 \pmod{17}$, we can solve these linear congruences to get particular solutions $x = 1$, $y = 4$, and $z = 8$. (These solutions can be found "mentally"; e.g., to solve $15z \equiv 1 \pmod{17}$, think of it as $(-2)z \equiv 18 \pmod{17}$.) So we get 187, 408, and 528 as our fundamental solutions. (Each is congruent to 1 modulo one of the three moduli and congruent to 0 modulo the other two moduli.) Finally, to solve the original problem, we take the requisite linear combination of the three fundamental solutions: $(5)(187) + (4)(408) + (3)(528) = 4151$. So the general solution is $x \equiv 4151 \pmod{1122}$ which reduces to $785 \pmod{1122}$.

(d) First we transform the given congruences so that the coefficients on the left-hand side disappear. $2x \equiv 1 \pmod 5$ is tantamount to $x \equiv 3 \pmod 5$. $3x \equiv 9 \pmod 6$ is tantamount to $x \equiv 3 \pmod 2$ or $x \equiv 1 \pmod 2$. $4x \equiv 1 \pmod 7$ is tantamount to $x \equiv 2 \pmod 7$. And $5x \equiv 9 \pmod{11}$ is tantamount to $x \equiv 4 \pmod{11}$. For this we'll use a hybrid method. The moduli are 2, 5, 7, and 11. We have $5 \times 7 \times 11 \equiv 1 \pmod 2$, $2 \times 7 \times 11 \equiv 4 \pmod 5$, $2 \times 5 \times 11 \equiv 5 \pmod 7$, $2 \times 5 \times 7 \equiv 4 \pmod{11}$. Solving the congruences $1a \equiv 1 \pmod 2$, $4b \equiv 3 \pmod 5$, $5c \equiv 2 \pmod 7$,

and $4d \equiv 4 \pmod{11}$, we get $a = 1$, $b = 2$, $c = -1$, and $d = 1$, so we obtain $x = (1)(5\times7\times11)+(2)(2\times7\times11)+(-1)(2\times5\times11)+(1)(2\times5\times7) = 653$ as a particular solution. The general solution is $x \equiv 653 \pmod{2\times5\times7\times11 = 770}$.

2. Let S, C, and F be a square, a cube, and a fourth power; if we denote the three (unknown) consecutive integers by $x - 1$, x, and $x + 1$, we see that we want to solve the congruences $x \equiv 1 \pmod{S}$, $x \equiv 0 \pmod{C}$, and $x \equiv -1 \pmod{F}$. In order to make it possible for x to satisfy these simultaneously, we will take S, C, and F relatively prime. Specifically, we take $S = 5^2 = 25$, $C = 3^3 = 27$, and $F = 2^4 = 16$. Let's solve this one by the method of progressive substitution: Since $x \equiv 1 \pmod{25}$, we can write $x = 25i + 1$. So $x \equiv 0 \pmod{27}$ becomes $25i + 1 \equiv 0 \pmod{27}$, or $-2i + 1 \equiv 0 \pmod{27}$, or $i \equiv 14 \pmod{27}$. So now we have $i = 27j + 14$ and $x = 25(27j + 14) + 1 = 675j + 351$. Finally, we must solve $675j + 351 \equiv -1 \pmod{16}$. Reducing this mod 16, we get $3j + 15 \equiv -1 \pmod{16}$, which has the handy solution $j \equiv 0 \pmod{16}$. So $x = 351$ will do the job. That is, the consecutive integers 350, 351, and 352 have the desired properties (being divisible by 25, 27, and 16, respectively).

Note: Some of you may have been tempted to let the square, cube, or fourth power be equal to 0 or 1. But in all such problems, you should think about the connection between the question that has been asked and the material that it relates to. Clearly this problem is designed to test one's ability to solve simultaneous congruences; letting the moduli be 0 or 1 makes the problems simpler, but your common sense should lead you to be suspicious of such simplifications. If something looks too good to be true, it probably is!

3. Letting x denote the number of eggs in the basket, we represent the constraints as $x \equiv 1 \pmod{2}$, $x \equiv 2 \pmod{3}$, $x \equiv 3 \pmod{4}$, $x \equiv 4 \pmod{5}$, $x \equiv 5 \pmod{6}$, and $x \equiv 0 \pmod{7}$. Note that -1 solves all of the congruences other than the last. So we are looking for a number that is congruent to -1 modulo 60 (the lcm of 3, 4, 5, and 6) and is a multiple of 7. That is, we want $60k - 1$ to be a multiple of 7. Since $60k - 1 \equiv 4k - 1 \pmod{7}$, we can get a solution to $60k - 1 \equiv 0 \pmod{7}$ by taking $k = 2$, yielding the solution $120 - 1 = 119$, which is the smallest possible number of eggs in the basket. (The general solution is $x \equiv 119 \pmod{420}$.)

4. Suppose $x \equiv a \pmod{n}$. Then $n|x-a$, that is $x-a = mn$ for some m. If m is even, we can write $m = 2k$, so that $x-a = 2kn$, whence $2n|x-a$ and $x \equiv a$

(mod $2n$). If m is odd, we can write $m = 2k + 1$, so that $x - a = 2kn + n$, whence $2n|x - a - n$ and $x \equiv a + n \pmod{2n}$.

5. The second congruence reduces to $2x \equiv 1 \pmod 3$. All three congruences may first be solved separately, yielding $x \equiv 4 \pmod 5$, $x \equiv 2 \pmod 3$, and $x \equiv 3 \pmod 7$. Thus we seek a number that is $\equiv -1 \pmod{15}$ and $\equiv 3 \pmod 7$. We can solve as in the earlier problems to obtain $x \equiv 59 \pmod{105}$. So (using the results of the previous problem) we see that the incongruent solutions modulo 210 are 59 and 164.

6. Multiplying the first equation by 4 (modulo 7), we get $4x + y \equiv 5 \pmod 7$. Subtracting this from the second of the two given congruences, we get $4y \equiv 1 \pmod 7$. Solving this, we get $y \equiv 2 \pmod 7$. Substituting this into the first of the two given congruences, we get $x + 4 \equiv 3 \pmod 7$, whence $x \equiv 6 \pmod 7$. So our general solution is $x \equiv 6 \pmod 7$, $y \equiv 2 \pmod 7$. Or, if you prefer, $x = 7j + 6$, $y = 7k + 2$.

7. Note that every prime other than 2 and 5 ends in 1, 3, 7, or 9. Note also that any product of numbers ending in 1 or 9 also ends in 1 or 9. Now suppose that there were only finitely many primes ending in 3 or 7. Let P be the product of all those primes, not including 3 (the reason for this exclusion will become clear below), and consider the number $N = 10P + 3$. Since N is divisible by neither 2 nor 5, all of its prime factors must end in 1, 3, 7, or 9. If they all ended in 1 or 9, then their product N would also have to end in 1 or 9, contradicting the fact that N ends in 3. Hence at least one prime factor p of N must end in a 3 or a 7. But such a prime would also have to be a prime factor of $10P$. Hence p would have to divide $N - 10P = 3$. But this is a contradiction, since we excluded 3 from the factors of P. Hence there cannot be only finitely many primes ending in 3 or 7.