Math 567, Assignment #6: Solutions

1. The sum of two multiplicative functions need not be multiplicative. For instance, $f(n) = n$ is multiplicative, but $g(n) = f(n) + f(n) = 2n$ is not (witness the fact that $g(6) = 12 \neq 24 = g(2)g(3)$). On the other hand, suppose f and g are multiplicative. Then their product F has the property that for all m and n relatively prime, $F(mn) = f(mn)g(mn) = f(m)f(n)g(m)g(n) = f(m)g(m)f(n)g(n) = F(m)F(n)$, which is to say that F is multiplicative.

2. Note that $720 = 2^4 3^2 5^1$.

(a) The sum of the square divisors of 2^4 is $1 + 2^2 + 2^4 = 21$, the sum of the square divisors of 3^2 is $1 + 3^2 = 10$, and the sum of the square divisors of 5^1 is 1. Hence the sum of the square divisors of 720 is $21 \times 10 \times 1 = 210$.

(b) We know that for m, n relatively prime, the divisors of mn are precisely the products de where d is a divisor of m and e is a divisor of n. Hence the squares of the divisors of mn are exactly the numbers that can be written as the product of the square of a divisor of m and the square of a divisor of n. It follows that the function is multiplicative. For $n = 2^4$, the function takes the value $1 + 2^2 + 2^4 + 2^6 + 2^8$, or 341. For $n = 3^2$, the sum of the squared divisors is $1 + 3^2 + 3^4$, or 91. For $n = 5^1$, the sum of the squared divisors is $1 + 5^2 = 26$. Hence the sum of the squared divisors of 720 is $341 \times 91 \times 26 = 806806$.

(c) For m, n relatively prime and k a squarefree divisor of mn, let $e = \gcd(k, m)$ and $f = \gcd(k, n)$. Since $e|k$ and k is squarefree, e is squarefree, and likewise for f. Conversely, the product of any two squarefree divisors of m and n (respectively) will be a squarefree divisor of mn. Hence the function is multiplicative. For 2^4, the function takes on the value $1 + 2 = 3$; for 3^2, the function takes on the value $1 + 3 = 4$; and for 5^1, the function takes on the value $1 + 5 = 6$. Hence the sum of the square-free divisors of 720 is $3 \times 4 \times 6 = 72$.

3. First, let us extend problem 1 by noting that if f, g are multiplicative functions, and g is non-vanishing, then the function $F = f/g$ is multiplicative as well: for m, n relatively prime, $F(mn) = f(mn)/g(mn) = f(m)f(n)/g(m)g(n) = (f(m)/g(m))(f(n)/g(n)) = F(m)F(n)$. Hence the multiplicativity of μ and τ implies the multiplicativity of μ^2 (since the product of two multiplicative functions is multiplicative), of μ^2/τ (since the quotient of two multiplicative functions is multiplicative), and of the function

$F(n)$ defined in the problem (by Theorem 6.4 in Burton). Therefore, it suffices to find a formula for $F(n)$ when n is a power of a prime.

Let's start by looking at $f(n) = \mu^2(n)/\tau(n)$. Putting $n = p^k$, we have $\mu(n) = 0$ unless $k \leq 1$. For $k = 0$, we have $n = p^0 = 1$ and $F(n) = 1$. For $k = 1$, we have $n = p$ and $\mu(n) = -1$ and $\tau(n) = 2$, so $\mu^2(n)/\tau(n) = +1/2$. Now, using $F(n) = \sum_{d|n} f(d)$, we find that for all $k \geq 1$, $F(p^k) = 1 + 1/2 + 0 + 0 + ... = 3/2$.

Finally, using the fact that F is multiplicative, we can write $F(n) = (3/2)^k$, where k is the number of distinct prime factors of n. Thus for example $F(720) = (3/2)^3$, since the distinct prime factors of 720 are 2, 3, and 5.