Math 832, Assignment #1

(questions 1–6 due February 10;
questions 7-11 due February 12)

All references are to Durrett.

1. Page 284, ex. 1.3

2. Page 284, ex. 1.6

3. Page 285, ex. 1.8

4. Page 290, ex. 2.6, part (i). Also, determine the function $h(x)$ when the Markov chain is unbiased random walk on 1,2,3,4,5 (i.e., $p_{1,2} = p_{5,4} = 1$, $p_{2,1} = p_{2,3} = p_{3,2} = p_{3,4} = p_{4,3} = p_{4,5} = \frac{1}{2}$) and where $A = \{1\}$, $B = \{5\}$.

5. Page 290, ex. 2.10, part (i). Also, determine the function $g(x)$ when the Markov chain is as in problem 4.

6. Let S be a finite set (for simplicity) and Ω be a probability space. Suppose the random variables $X_1, X_2, ..., X_n$ (functions on Ω) are a (finite-duration, not-necessarily-temporally-homogeneous) Markov chain relative to the usual filtration generated by the random variables themselves. Suppose A is some subset of S such that $P(X_n \in A) > 0$. Let $\Omega' = \{\omega : X_n(\omega) = A\}$, and let $X_1', X_2', ..., X_n'$ be the restrictions of the respective X_i's to the set Ω'. Is $X_1', X_2', ..., X_n'$ necessarily a Markov chain? If the original chain was temporally homogeneous, must the new chain be temporally homogeneous? In each case, give a proof or a counterexample.

7. If a Markov chain on countably many states is transient, so that $N_x(y)$ (the expected number of visits to y given that one starts at x) is almost surely finite and has finite expected value, can $N_x(y)$ have infinite variance?

8. Page 292, ex. 3.2

9. Page 292, ex. 3.3 (be sure to handle convergence issues correctly)

10. Is the independent product of two irreducible Markov chains irreducible? (Here the product is the Markov chain whose state space is the Cartesian product of the states spaces of the two original chains, and whose transition probability matrix is the tensor product of the transition probability matrices of the two original chains; that is, $p((x_1, x_2), (y_1, y_2)) = p(x_1, y_1)p(x_2, y_2)$.)

(more)

11. (a) How many times on average does one need to toss a fair coin before one first sees the sequence THTH?

(b) Same as part (a), but with the sequence HTHH.

(c) Which is one more likely to observe first: THTH or HTHH? How much more likely?

(Note: You will probably find exercises 2.6 and 2.10 on page 290 of Durrett helpful. Also, note that the answer to part (c) does *not* follow from the answers to parts (a) and (b).)