Math 832, Assignment #2

(due February 24)

1. The heat bath Markov chain: Suppose we have a recurrent Markov chain on a finite state space S and a non-negative function f on S with the property that, whenever $p(x,y)$ and $p(x,y')$ are both non-zero, $p(x,y)/p(x,y') = f(y)/f(y')$. Let $Z = \sum_{x\in S} f(x)$. Show that the probability distribution μ defined by $\mu(x) = f(x)/Z$ is an stationary measure for $p(.,.)$. [**NOTE: This problem is erroneous.**]

2. Let $p(\cdot,\cdot)$ be a transition probability function on a countable state-space S, μ be a probability measure on S that is stationary for p, and q be a transition probability function dual to p in the sense that

$$\mu(x)q(x,y) = \mu(y)p(y,x)$$

for all x, y. (Note that this generalizes the definition Durrett gives in Exercise 4.1, since it permits $\mu(x)$ to vanish.)

(a) Verify that μ is a stationary measure for q.

(b) Suppose $x \in S$ is a recurrent state for the transition probabilities $p(\cdot,\cdot)$. Is $x \in S$ necessarily a recurrent state for the transition probabilities $q(\cdot,\cdot)$?

(c) Suppose every state is recurrent for the transition probabilities $p(\cdot,\cdot)$. Is the same necessarily true for the transition probabilities $q(\cdot,\cdot)$?

3. Given two probability measures μ and ν on a countable set S define the distance between them as $d(\mu,\nu) = \sum_{x\in S} |\mu(x) - \nu(x)|$. (Note that this is just twice the usual total variation norm.)

(a) Show that for all transition probability functions $p(\cdot,\cdot)$, $d(\mu p, \nu p) \le d(\mu,\nu)$.

(b) Now assume S is finite and $p(x,y) > 0$ for all x, y. Show that there exists a constant $c < 1$ such that $d(\mu p, \nu p) \le c\,d(\mu,\nu)$. Conclude from this that a stationary distribution π exists and that $p^n(x,y) \to \pi(y)$.

4. Suppose $\{X_n : n = 0, 1, 2, \ldots\}$ is a Markov chain on a finite state space S. Let A be any subset of S and let $\tau_0, \tau_1, \tau_2, \ldots$ be the return times to A defined by $\tau_0 = \inf\{n : X_n \in A\}$ and $\tau_{k+1} = \inf\{n > \tau_k : X_n \in A\}$. Let $Y_n = X_{\tau_n}$. Prove or give a counterexample: $\{Y_n\}$ is a Markov chain.

5. Fix n and k, $0 < k < n$. Let S be the set of all mappings from Z/nZ (the integers mod n) to $\{0, 1\}$ that take the value 1 exactly k times. (We'll call this a "circular bit-string"; you may wish to think of this as a circular arrangement of k lords and $n-k$ ladies, of the sort discussed in lecture during the first two weeks.)

(a) Suppose our Markov chain step is as follows: if the current state contains the substring 1,0 exactly m times (as it must for some $m \geq 1$), then one randomly picks *one* of those m occurrences and replaces it by 0,1. Show that this Markov chain is irreducible and recurrent, and find the (unique) stationary probability measure.

(b) Suppose our Markov chain step is as follows: if the current state contains the substring 1,0 exactly m times (as it must for some $m \geq 1$), then for each of the m occurrences one independently tosses an unbiased coin and leaves the substring alone (resp. exchanges the 0 and 1) if the coin comes up heads (resp. tails). Show that this Markov chain is irreducible and recurrent, and find the (unique) stationary probability measure.