Math 832, Assignment #3

(due March 23)

1. "Sampling from the stationary distribution of a non-stationary Markov chain": Suppose $X_0, X_1, X_2, ...$ is a Markov chain with state space $S = \{1, 2\}$ and transition probabilities $p(1,1) = 1-p$, $p(1,2) = p$, $p(2,1) = q$, $p(2,2) = 1-q$, with $0 < p < 1$ and $0 < q < 1$. Let π denote the stationary distribution on S. Do *not* assume that X_0 is governed by π (that is, do not assume that $\text{Prob}(X_0 = i) = \pi(i)$). Find a stopping time N with $E(N) < \infty$ such that X_N *is* governed by π (and of course verify that N does indeed satisfy these properties). Your definition of N should not involve p or q in any way. (Hint: Compose random maps that respect $p(.,.)$, and use coupling from the past. The random maps can be obtained from the observed values of $X_0, X_1, X_2, ...$)

2. "One-dimensional Ising model": Let Ω be the set of length-n strings of $+1$'s and -1's, and define the energy $E(s)$ of a string $s = (s_1, s_2, ..., s_n) \in \Omega$ as $\sum_{k=1}^{n-1} -s_k s_{k+1}$. Let μ be the probability distribution on Ω that gives each state s probability proportional to $q^{-E(s)}$ for some fixed $q > 1$. (Note: μ is the Gibbs distribution for the Ising model at absolute temperature $1/\log q$; taking $q < 1$ would correspond to having a "negative temperature".) Let $X_i(s) = s_i$. Show that the random variables $X_1, X_2, ..., X_n$ form a Markov chain.

3. Let Ω be the set of sequences $s = (s_{-n}, s_{-n+1}, ..., s_0, ..., s_n)$ with $s_i \in \{0, 1\}$ for all i ($-n \le i \le n$), and let $X_i(s)$ denote s_i.

(a) Show that for $a, b \in \{0, 1\}$ there is a unique probability measure on Ω such that $X_{-n} = a$ a.s., $X_n = b$ a.s., and for all i with $-n < i < n$,

$$\text{Prob}(X_i = 1 | X_{-n} = x_{-n}, ..., \widehat{X_i = x_0}, ..., X_n = x_n) = \begin{cases} \frac{1}{2} & \text{if } x_{i-1} = x_{i+1} = 0, \\ 0 & \text{otherwise.} \end{cases}$$

(Here "$\widehat{..}$" denotes a term that is omitted.) That is, show that the "inward transition function" given above determines a unique (1-dimensional) Markov random field.

(b) Is $X_{-n}, ..., X_n$ a Markov chain?

(c) What happens to $\text{Prob}(X_0 = 1)$ as $n \to \infty$?

4. Let S be a finite set, and for all i, j in S with $i \neq j$ let the rates $r_{i,j}$ be non-negative numbers.

(a) Prove the existence of a stationary measure for the Markov process with state space S and rates $r_{i,j}$.

(b) Find a necessary and sufficient condition for the uniqueness of the stationary measure (and prove its validity, of course).

5. Fix n and k, $0 < k < n$. Let S be the set of all mappings from Z/nZ (the integers mod n) to $\{0, 1\}$ that take the value 1 exactly k times. (This is the setting of problem 4 from the previous assignment, but now we will do our updates in continuous time.) Suppose that for all states $s, t \in S$, the rate $r_{s,t}$ at which the system jumps from state s to state t is 0 unless t is obtained from s by replacing a single substring 1,0 in s with the substring 0,1 in exactly the same location, in which case $r_{s,t} = 1$. Find a stationary measure for this Markov process (unique by Problem 4).