Math 832, Assignment #4

(due April 13)

In problems 1, 2, 3, and 5, assume that $(X, \mathcal{F}, \mu, T)$ is a measurable dynamical system with T invertible, and that f is a measurable real-valued function on X.

1. Show directly (i.e., without using the ergodic theorem) that for all constants c, the set

$$\{x \in X : (\forall \epsilon > 0)(\exists n_0)(\forall n \geq n_0) : \\ \alpha - \epsilon < (f(x) + f(Tx) + f(T^2x) + \ldots + f(T^{n-1}x))/n < \alpha + \epsilon\}$$

is invariant (i.e. it coincides with its T-preimage modulo a set of measure 0).

2. Assume that the function $f^*(x) = \lim_{n\to\infty} \frac{1}{n}(f(x)+f(Tx)+\cdots+f(T^{n-1}x))$ is well-defined almost everywhere; show that it is invariant (i.e., $f^*(x) = f^*(Tx)$ a.e.).

3. Show that the class $\mathcal{I}$ of invariant sets in $\mathcal{F}$ is a σ-algebra, and that for any measurable function f, the following three conditions are equivalent: (1) the function f is $\mathcal{I}$-measurable; (2) the function f satisfies $f \circ T = f$ a.s.; and (3) there exists a measurable function g such that $g \circ T = g$ (everywhere) and such that $f = g$ a.s. (Hint: The steps given in Durrett's Exercise 1.2 may give you some ideas for how to do this.)

4. Let $X_0, X_1, \ldots$ be some real-valued stationary stochastic process. Turn it into a dynamical system by letting Ω be product of countably many copies of the reals, letting $\mathcal{F}$ be the σ-algebra $\sigma(X_0, X_1, \ldots)$, etc. Show that the tail σ-algebra $\cap_n \sigma(X_n, X_{n+1}, \ldots)$ contains all the invariant sets in the dynamical system (modulo sets of measure 0).

5. Given a dynamical system $(X, \mathcal{F}, \mu, T)$, show that the following assertions are equivalent: (1) the system is not ergodic; (2) there exist invariant sets $X_1, X_2 \subset X$ with $X_1 \cap X_2 = \phi$, $X_1 \cup X_2 = X$, and $0 < \mu(X_1), \mu(X_2) < 1$; (3) there exist mutually singular probability measures μ_1, μ_2 on $(X, \mathcal{A})$ that are invariant under T such that $\mu = \alpha\mu_1 + (1-\alpha)\mu_2$ for some $0 < \alpha < 1$.