Math 832, Assignment #5

(due May 16)

1. (Durrett, ex. 7.1.3): Fix t and let $\Delta_{m,n} = B(tm2^{-n}) - B(t(m-1)2^{-n})$. Compute

$$E\left(\sum_{m\le 2^n} \Delta_{m,n}^2 - t\right)^2$$

and use Borel-Cantelli to conclude that $\sum_{m\le 2^n} \Delta_{m,n}^2 \to t$ a.s. as $n\to\infty$.

2. (Durrett, ex. 7.2.1): Let $T_0 = \inf\{s>0 : B_s = 0\}$ and let $R = \inf\{t>1 : B_t = 0\}$. ($R$ is for right or return.) Use the Markov property at time 1 to get

$$P_x(R > 1+t) = \int p_1(x,y) P_y(T_0 > t)\, dy\,.$$

3. (Durrett, ex. 7.2.3): Prove that if $a<b$, there is a.s. a local maximum of B_t in (a,b). Conclude that the set of local maxima of B_t is a.s. a dense set.

4. (a) Suppose that $B_1(t)$ and $B_2(t)$ are independent Brownian motions. For what non-zero real numbers c (if any) is it the case that $(B_1(t)+B_2(t))/c$ is a Brownian motion?

(b) Suppose that $B(t)$ is a Brownian motion started at 0. Put $B_1(t) = B(t)$ and $B_2(t) = B(t+1) - B(1)$ for all $t\ge 0$. For what non-zero real numbers c (if any) is it the case that $(B_1(t)+B_2(t))/c$ is a Brownian motion?

5. (a) Take n large, and consider a sequence of i.i.d. random variables $X_1,\ldots,X_{2n}$ with $P(X_i=+1) = P(X_i=-1) = \frac{1}{2}$ for all i. For $1\le k\le 2n$, put $S_k = X_1 + X_2 + \ldots + X_k$. Now condition this whole stochastic process on the event $S_{2n} = X_1+\ldots+X_{2n} = 0$. Compute the conditional mean and conditional variance of $S_m/\sqrt{2n} - S_k/\sqrt{2n}$ for $k<m$. Compute also the limits of these quantities as $n\to\infty$ with $k/2n\to\alpha$ and $m/2n\to\beta$, where $0\le\alpha\le\beta\le 1$.

(b) For $B(t)$ a Brownian motion started at 0, define $C(t) = B(t) - tB(1)$ for $0\le t\le 1$. Find the mean and variance of $C(\beta) - C(\alpha)$ for $0\le\alpha\le\beta\le 1$.

(c) Formulate a conjecture based on what you learned from parts (a) and (b). A formal proof is not requested, but explain your reasons for believing the conjecture and/or the points that you feel least certain about.