Math 832, Assignment #4: Solutions

1. Call the set A_α; it can be written more succinctly as $\{x \in X : \frac{1}{n}(f(x) + f(T^2x) + \cdots + f(T^{n-1}x)) \to \alpha \text{ as } n \to \infty\}$. Given $x \in A_\alpha$, we have $\frac{1}{n}(f(Tx) + f(T^2x) + \cdots + f(T^nx)) = \frac{n+1}{n}(\frac{1}{n+1}(f(x) + \cdots + f(T^nx))) - \frac{1}{n}f(x) \to 1 \cdot \alpha - 0 \cdot f(x) = \alpha$, so $Tx \in A_\alpha$.

2. Suppose B is a set of μ-measure 0 such that $f^*(x)$ is well-defined on $X \setminus B$. Then, taking $\alpha = f^*(x)$, we have $x \in A_\alpha$, so by exercise 1, $Tx \in A_\alpha$ as well, whence $f^*(Tx) = \alpha = f^*(x)$. Since this holds for all x outside B, $f^* = f$ a.e.

3. First, note that if $\mu(T^{-1}A \triangle A) = 0$ and $\mu(T^{-1}B \triangle B) = 0$ then $\mu(T^{-1}(A \cup B) \triangle (A \cup B)) = 0 = \mu(T^{-1}(A \cap B) \triangle (A \cap B))$; hence $A, B \in \mathcal{I}$ implies $A \cup B,\ A \cap B \in \mathcal{I}$. In fact, since a countable union of null-sets is null, we have $\mu(T^{-1}C \triangle C) = 0$ whenever C is a countable union $C = \bigcup_{k=1}^{n} A_k$ of sets $A_k \in \mathcal{I}$. Hence $\mathcal{I}$ is a σ-algebra.

(1) implies (2): For every interval (r, s) with r, s rational and $r < s$, let $A_{r,s} = \{x : r < f(x) < s\}$. Every x with $f(Tx) \neq f(x)$ must lie in $T^{-1}A_{r,s}| \triangle A_{r,s}$ for some r, s (any interval (r, s) containing x but not Tx, or vice versa, will do). Hence $\{x : f(Tx) \neq f(x)\}$, as a subset of a countable union of sets $T^{-1}A_{r,s} \triangle A_{r,s}$ of measure 0, must have measure 0 itself.

(2) implies (3): For x lying in the set $\{x : f(T^nx) = f(x) \text{ for all } x \in Z\}$, put $g(x) = f(x)$, and for all x lying in the complementary set (whose measure is zero), put $g(x) = c$, where c is any constant. Then $g \circ T = g$ everywhere and $f = g$ on a set of full measure.

(3) implies (1): $\mathcal{I}$ is generated by the pre-images of all the Borel sets; but since the σ-algebra of Borel sets is generated by the open intervals in $\mathbf{R}$, $\mathcal{I}$ is also generated by the pre-images of the open intervals. Hence, to show that f is $\mathcal{I}$-measurable, it suffices to prove that $f^{-1}((r, s))$ is invariant. Suppose $x \in f^{-1}((r, s))$, so that $f(x) \in (r, s)$. Then, unless x belongs to the bad set $\{x : f(x) \neq g(x)\}$ or the pre-image (T^{-1}) of that set, both of which have measure 0, we have $f(Tx) = g(Tx) = g(x) = f(x) \in (r, s)$, where $Tx \in f^{-1}((r, s))$. It follows that the set $f^{-1}((r, s))$ is invariant. as was to be shown.

4. We will show that for any (real-valued) stationary stochastic process $(X_0, X_1, X_2, \ldots)$, the tail $\bigcap_{n=0}^{\infty} \sigma(X_n, X_{n+1}, \ldots)$ contains all invariant sets. Let $(\Omega, \mathcal{F}, \mu, T)$ be the associated dynamical system with $\Omega = \mathbf{R}^{\mathbf{N}}$ and

$X_n(\omega) = f(T^n\omega)$. Suppose $E \subseteq \Omega_A$ is an invariant set. Then the Borel set $E' = \bigcup_{n=0}^{\infty} T^{-n}E$ is *strictly* invariant (i.e., $T^{-1}E'$ and E' are equal as sets). Then we have $E' = T^{-n}E' \in \sigma(X_n, X_{n+1}, \ldots)$ for all n, so E' is in the tail σ-field. Since $\mu(E \triangle E') = 0$, we see that every invariant set E is in the tail, modulo a set of measure 0.

5. Suppose the system is not ergodic (property (1)), so that there exists an invariant set A with $0 < \mu(A) < 1$. Then, setting $X_1 = A$ and $X_2 = X \setminus A$, we verify property (2).

Now, assuming only property (2), we can define $\mu_i(\cdot) = \mu(\cdot|X_i)$ for $i = 1, 2$. Since $\mu_i(X_j) = \delta(i,j)$, the probability measures μ_1 and μ_2 are mutually singular. Putting $\alpha = \mu(X_1) = 1 - \mu(X_2)$, we have $0 < \alpha < 1$ and $\mu(A) = \mu(A \cap X_1) + \mu(A \cap X_2) = \alpha\mu_1(A) + (1-\alpha)\mu_2(A)$ as claimed. Furthermore, for all $A \in \mathcal{F}$ we have $\mu_i(T^{-1}A) = \mu(T^{-1}A \cap X_i)/\mu(X_i) = \mu(T^{-1}A \cap T^{-1}X_i)/\mu(X_i) = \mu(T^{-1}(A \cap X_i))/\mu(X_i) = \mu(A \cap X_i)/\mu(X_i) = \mu_i(A)$, so that μ_i is invariant, verifying property (3).

Lastly, assuming only property (3), we know from the mutual singularity of μ_1 and μ_2 that we can find supporting sets $S_1, S_2 \in \mathcal{F}$ such that $\mu_i(X_j) = \delta_{i,j}$. (Note that S_1, S_2 need not satisfy $S_1 \cap S_2 = \phi$ or $S_1 \cup S_2 = X$.) Then $\mu(S_1) = \alpha\mu_1(S_1) + (1-\alpha)\mu_2(S_2) = \alpha \in (0,1)$, so $0 < \mu(S_1) < 1$. On the other hand, I claim that S_1 is invariant. For, suppose $\mu(S_1 \triangle T^{-1}S_1) > 0$. Then $\mu(T^{-1}S_1) = \mu(S_1 \cap T^{-1}S_1)/\mu(S_1) < 1 = \mu_1(S_1)$, contradicting the assumption that μ_1 is invariant. This verifies property (1).