Math 832, Assignment #5: Solutions

1. The scaling relationship (1.2) implies

$$\Delta_{m,n} \stackrel{d}{=} 2^{-n/2}\Delta_{1,0}, \tag{1}$$

while the definition of Brownian motion shows $E\Delta_{1,0}^2 = t$ and $E(\Delta_{1,0}^2 - t)^2 = C < \infty$. Using (1) and the definition of Brownian motion, it follows that if $k \neq m$ then $\Delta_{k,n}^2 - t2^{-n}$ and $\Delta_{m,n}^2 - t2^{-n}$ are independent and have mean 0 so

$$E\left(\sum_{1\le m\le 2^n}(\Delta_{m,n}^2 - t2^{-n})\right)^2 = \sum_{1\le m\le 2^n} E(\Delta_{m,n}^2 - t2^{-n})^2 = 2^n C 2^{-2n}$$

where in the last equality we have used (1) again. The last result and Chebyshev's inequality imply

$$P\left(\left|\sum_{1\le m\le 2^n}\Delta_{m,n}^2 - t\right| \ge 1/n\right) \le Cn^2 2^{-n}.$$

The right hand side is summable so the Borel-Cantelli lemma implies

$$P\left(\left|\sum_{1\le m\le 2^n}\Delta_{m,n}^2 - t\right| \ge 1/n \text{ infinitely often}\right) = 0.$$

2. Let $Y = 1_{(T_0>t)}$ and note that $T_0 \circ \theta_1 = R - 1$ so $Y \circ \theta_1 = 1_{(R>1+t)}$. Using the Markov property gives $P_x(R > 1+t|\mathcal{F}_1) = P_{B_1}(T_0 > t)$. Taking expected value now and recalling $P_x(B_1 = y) = p_1(x,y)$, we get $P_x(R > 1+t) = \int p_1(x,y)P_y(T_0 > t)\, dy$.

3. (2.6), (2.7), and the Markov property imply that with probability 1 there are times $s_1 > t_1 > s_2 > t_2 > \ldots$ that converge to a so that $B(s_n) = B(a)$ and $B(t_n) > B(a)$. In each interval (s_{n+1}, s_n) there is a local maximum. Hence (a,b) contains a local maximum, and the density claim follows immediately.

4. (a) Yes, $c = \pm\sqrt{2}$ will do (but no other constants). For any non-zero constant c, $X_t = (B_1(t) + B_2(t))/c$ is a Gaussian process (that is, for each particular t, the distribution is a normal distribution with mean 0), and the

increments are independent. For all $x, t \geq 0$, independence of B_1 and B_2 gives $E((X(s+t)-X(s))^2) = \frac{1}{c^2}[E(B_1(s+t)-B_1(s))^2 + E(B_2(s+t)-B_2(s))^2] = \frac{1}{c^2}(2s)$, which equals s iff $c = \pm\sqrt{2}$.

(b) No constant will make the process (call it $X(t)$) Brownian, because beyond time 1 the increments cease to be independent. We check this by calculating the covariance between $X(1)-X(0)$ and $X(2)-X(1)$, taking $c = 1$ for convenience since this scaling does not affect the issue of independence: $X(1)-X(0) = B(2)$ and $X(2)-X(1) = B(3)-B(1)$, so $E((X(1)-X(0))(X(2)-X(1))) = EB(2)B(3) - EB(2)B(1) = \min(2,3) - \min(2,1) = 2-1 = 1 \neq 0$.

5. (a) The conditional distribution on $(X(1), \ldots, X(n))$ is the uniform distribution on all sequences of $+1$'s and -1's consisting of equal number of each. This distribution (tacit hereafter) satisfies $EX_i = 0$ and $EX_i^2 = 1$ for all i, and $EX_iX_j = -\frac{1}{2n-1}$ for all $i \neq j$. Hence the conditional mean of $S_m - S_k = X_{k+1} + \ldots + X_m$ is zero, and the conditional variance is $E(X_{k+1} + \ldots + X_m)^2 = \sum_{i=k+1}^{m} E(X_i)^2 + \sum_{k+1 \leq i,j \leq m;\ i \neq j} EX_iX_j = (m-k) - (m-k)(m-k-1)/(2n-1)$. Approximating k and m by $2n\alpha$ and $2n\beta$ respectively, we see that this expression is close to $2n(\beta-\alpha) - 2n(\beta-\alpha)^2$. Hence, if we divide the random variables by $\sqrt{2n}$, the variance of $S_m/\sqrt{2n} - S_k/\sqrt{2n}$ is seen to approach $(\beta-\alpha) - (\beta-\alpha)^2$ as $n \to \infty$.

(b) The expected value of $C(\beta) - C(\alpha)$ is $E(B(\beta) - \beta B(1) - B(\alpha) + \alpha B(1)) = 0$. The variance is $E(B(\beta) - B(\alpha) - (\beta-\alpha)B(1))^2 = E(B(\beta))^2 + E(B(\alpha))^2 + (\beta-\alpha)^2 E(B(1))^2 - 2EB(\alpha)B(\beta) - 2(\beta-\alpha)EB(\beta)B(1) + 2(\beta-\alpha)EB(\alpha)B(1) = \beta + \alpha + (\beta-\alpha)^2 - 2\alpha - 2(\beta-\alpha)\beta + 2(\beta-\alpha)\alpha = (\beta-\alpha) - (\beta-\alpha)^2$.

(c) One might conjecture that the pinned random walks analyzed in part (a) converge weakly to the modified Brownian motions analyzed in part (b). That is: By re-scaling time by n and space by $\sqrt{n}$, and by using linear interpolation, one can replace each sample path from the pinned random walk by a (piecewise-linear) continuous real-valued function $x(\cdot)$ on $[0,1]$ satisfying $x(0) = x(1) = 0$. Then the distribution on pinned random walk of length $2n$ gives rise to a distribution P_n on the space of such functions, and one might conjecture (correctly!) that the P_n's converge in distribution to the so-called "Brownian bridge" $B(t) - tB(1)$.