# Evolving cross sections of Ford spheres

*Illustrating Math* show-and-ask

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These slides are at
https://faculty.uml.edu/jpropp/ford-spheres.pdf.

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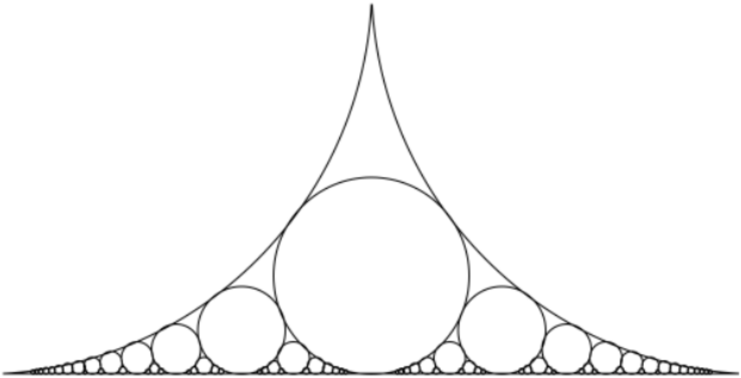
Not true! There can be geometry hiding here that comes into view if we give each element of the countable dense set $S \subseteq \mathbb{R}$ some real estate in the half-plane $\{(x, y) \mid y \geq 0\} \subseteq \mathbb{R}^2$, so that $S$ is the limit set of non-overlapping regions in the half-plane.

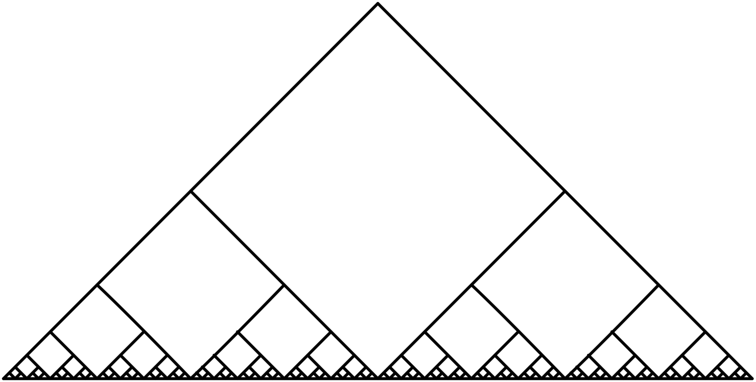
# Example: $\mathbb{Q}$

The set of rational numbers is the limit set of the Ford circles.



# Example: $\mathbb{Z}[1/2]$

The set of dyadic rationals (rationals whose denominators are powers of 2) is the limit set of the “Ford squares”.

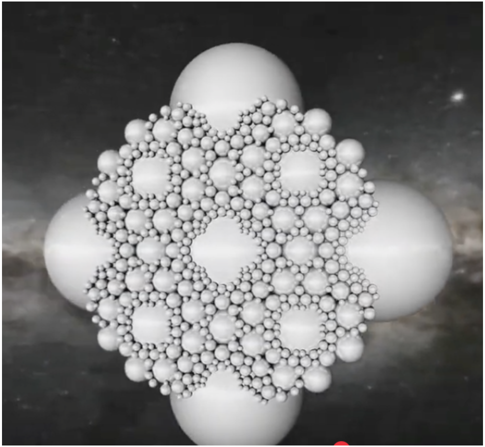


# Countable dense subsets of the complex plane

Likewise, even though the set of Gaussian rationals $\mathbb{Q}(i) = \{a + bi \mid a, b \in \mathbb{Q}\}$ and the set of Eisenstein rationals $\mathbb{Q}(\omega) = \{a + b\omega \mid a, b \in \mathbb{Q}\}$ (with $\omega = \frac{-1+\sqrt{-3}}{2}$) are **both countable dense subsets** of the complex plane,

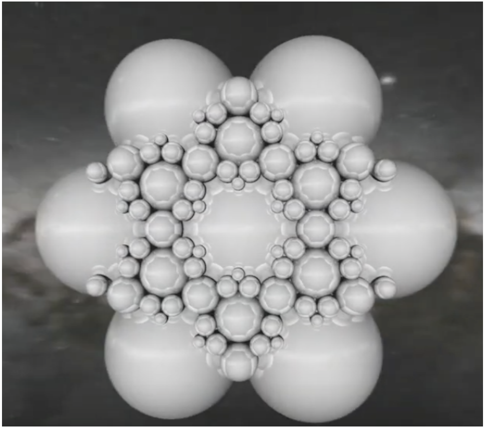
they are the limit sets of geometrically **different** fractal sphere-arrangements in 3-space ("Ford spheres" as described by Sam Northshield and others, and depicted in Sam Wells and Aidan Donahue's video "What are Ford spheres?").

# Ford spheres for $\mathbb{Q}(i)$ (from the video)



(the five big spheres correspond to 0 and the 4th roots of 1)

# Ford spheres for $\mathbb{Q}(\omega)$ (from the video)



(the seven big spheres correspond to 0 and the 6th roots of 1)

# Picture if you will:

Take $\zeta = i$ or $\zeta = \omega$.

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Identify the complex plane with the plane $\{(x, y, z) \mid z = 0\}$ in $\mathbb{R}^3$, and look at the Ford spheres in the upper half space $\{(x, y, z) \mid z \geq 0\}$ whose intersection with the limit-plane $z = 0$ is $\mathbb{Q}(\zeta)$.

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Let's try to visualize the three-dimensional fractal set $B_\zeta$ using two-dimensional cross-sections that change over time, as the cutting-plane $z = h$ approaches the limit-plane $z = 0$.

# The movie(s) I want to see

For $h > 0$ let $B_\zeta(h)$ be the intersection of the closed set $B_\zeta$ with the cutting-plane $z = h$ — a union of closed disks (a few large disks when $h$ is big, many small disks when $h$ is small).

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I want to watch it happen (both without and with continuous zooming).

# Disclaimer

I don't do research in this area.

I don't at this time want to learn how to create animations.

I just think these animations would be fun to watch.

And perhaps they would be of independent interest for math outreach, as cool examples of what it means for a countable set to be dense ("How can a set be all over the place but hardly anywhere?") or just as fan-service for fractal-philes.

Thanks! — jamespropp@gmail.com

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