Tiling lattices with sublattices

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Mirsky and Newman (after a conjecture of Erdős): If $\mathbf{Z}$ is written as a disjoint union of finitely many two-sided arithmetic progressions

$$a_1 + d_1\mathbf{Z},\ a_2 + d_2\mathbf{Z},\ \ldots,\ a_n + d_n\mathbf{Z}$$

with $n > 1$, then two of the d_i's must be equal.

That is, if we tile the 1-dimensional lattice $\mathbf{Z}$ by translates of sublattices of $\mathbf{Z}$, two of the sublattices must be the same.

I'll give a Fourier analysis proof and then show how it yields a generalization of the Mirsky-Newman result for tilings of higher-dimensional lattices by sublattices.

"Book proof" of Mirsky-Newman theorem (found by Mirsky and Newman, rediscovered by Davenport and Rado):

Write $\mathbf{N} = \{0, 1, 2, \ldots\}$ as a disjoint union of the sets $a_1+d_1\mathbf{N}, ..., a_n+d_n\mathbf{N}$ (adjusting the a_i's as needed) so that

$$\frac{1}{1-z} = \frac{z^{a_1}}{1-z^{d_1}} + \ldots + \frac{z^{a_n}}{1-z^{d_n}}.$$

Let $D = d_m > 1$ be the largest of the d_i's.

The associated term $\frac{z^{a_m}}{1-z^{d_m}}$ in the RHS has a pole at $\exp(2\pi i/D)$ but the LHS does not, so there must be another term $\frac{z^{a_j}}{1-z^{d_j}}$ in the RHS that cancels the pole, with D dividing d_j.

But $D \geq d_j$ by choice of D, so $d_j = D = d_m$. $\diamondsuit$

We prefer to work in the two-sided setting ($\mathbf{Z}$ instead of $\mathbf{N}$) and use (discrete) Fourier transforms instead of generating functions.

E.g., consider the tiling of $\mathbf{Z}$ by $2\mathbf{Z}$, $4\mathbf{Z}+1$, and $4\mathbf{Z}+3$.

(With generating functions this corresponds to the decomposition

$$\frac{1}{1-z} = \frac{1}{1-z^2} + \frac{z}{1-z^4} + \frac{z^3}{1-z^4}$$

where the last two terms on the RHS have poles at $z = i$ and at $z = -i$ that cancel each other.)

We take the discrete Fourier transforms of the indicator functions of the sets $\mathbf{Z}$, $2\mathbf{Z}$, $4\mathbf{Z}+1$, and $4\mathbf{Z}+3$.

$$
\begin{aligned}
1_{\mathbf{Z}}(n) &= 1^n \\
1_{2\mathbf{Z}}(n) &= (1/2)1^n + (1/2)(-1)^n \\
1_{4\mathbf{Z}+1}(n) &= (1/4)1^n + (-i/4)(i)^n \\
&\quad +(-1/4)(-1)^n + (i/4)(-i)^n \\
1_{4\mathbf{Z}+3}(n) &= (1/4)1^n + (i/4)(i)^n \\
&\quad +(-1/4)(-1)^n + (-i/4)(-i)^n
\end{aligned}
$$

Check that the coefficients of 1^n add up to 1 while the other coefficients cancel.

We write 1^n, i^n, $(-1)^n$, and $(-i)^n$ as $\exp(2\pi ikn)$ with $k = 0$, $1/4$, $1/2$, and $3/4$, respectively. Then the Fourier transform of $1_{4\mathbf{Z}+3}$ is the function that sends $0, 1/4, 1/2, 3/4$ to $1/4, i/4, -1/4, -i/4$ (respectively) and vanishes elsewhere, and similarly for the other sets.

For all k in $\mathbf{Q}/\mathbf{Z} \approx \mathbf{Q} \cap [0,1)$, let $\delta(k)$ be the function on $\mathbf{Q}/\mathbf{Z}$ that equals 1 at k and 0 everywhere else. Then

$$\begin{aligned}
\widehat{1_{\mathbf{Z}}} &= \delta(0) \\
\widehat{1_{2\mathbf{Z}}} &= (1/2)\delta(0) + (1/2)\delta(1/2) \\
\widehat{1_{4\mathbf{Z}+1}} &= (1/4)\delta(0) + (-i/4)\delta(1/4) \\
&\quad +(-1/4)\delta(1/2) + (i/4)\delta(3/4) \\
\widehat{1_{4\mathbf{Z}+3}} &= (1/4)\delta(0) + (i/4)\delta(1/4) \\
&\quad +(-1/4)\delta(1/2) + (-i/4)\delta(3/4)
\end{aligned}$$

The last two Fourier transforms have non-zero values at $1/4$ and $3/4$ that cancel each other (cf. the cancellation between $x/(1-x^4)$ and $x^3/(1-x^4)$ for the generating function approach).

Fourier proof of Mirsky-Newman theorem:

Write $\mathbf{Z} = \{0, 1, 2, \ldots\}$ as a disjoint union of the sets $A_1 = a_1 + d_1\mathbf{Z}$, ..., $A_n = a_n + d_n\mathbf{Z}$ so that

$$1_{\mathbf{Z}} = 1_{A_1} + \ldots + 1_{A_n}$$

whence

$$\widehat{1_{\mathbf{Z}}} = \widehat{1_{A_1}} + \ldots + \widehat{1_{A_n}}.$$

Let $D = \max(d_1, \ldots, d_n) = d_m > 1$.

$\widehat{1_{\mathbf{Z}}}$ vanishes at $k = 1/D$ but $\widehat{1_{A_m}}$ does not, so there must be another term $\widehat{1_{A_j}}$ that cancels it with D dividing d_j, and as before, we get $d_j = d_m$. $\diamondsuit$

This approach generalizes to tilings of $\mathbf{Z}^d$ by translates of sublattices of the form $L = a_1\mathbf{Z} \times \ldots \times a_d\mathbf{Z}$ for positive integers $a_1, \ldots, a_d$. We call these *straight* sublattices of $\mathbf{Z}^d$.

THEOREM: Given $n > 1$ translates of straight sublattices tiling $\mathbf{Z}^d$, two of the tiles must be translates of each other.

PROOF: Write the tiles as $L_i + \mathbf{v}_i$ with L_i a straight sublattice of $\mathbf{Z}^d$ and $\mathbf{v}_i \in \mathbf{Z}^d$, and let f_i be the indicator function of $L_i + \mathbf{v}_i$, so that $1_{\mathbf{Z}^d} = \sum_i f_i$.

Each f_i is periodic on $\mathbf{Z}^d$ and so can be written uniquely in the form $\mathbf{x} \mapsto \sum_{\mathbf{k}\in K} c_{\mathbf{k}} \exp(2\pi i \mathbf{k} \cdot \mathbf{x})$ where K (the "spectrum" of f) is a finite subset of $(\mathbf{Q} \cap [0, 1))^d$ and the $c_{\mathbf{k}}$'s are non-zero complex numbers.

(The map that send $\mathbf{k}$ to $c_{\mathbf{k}}$ and vanishes outside of K is the discrete Fourier transform $\hat{f}$ of f.)

For $L_i = a_1\mathbf{Z}\times \ldots \times a_d\mathbf{Z}$, K is $\{(r_1/a_1, \ldots, r_d/a_d) : \ 0 \leq r_i < a_i$ for $1 \leq i \leq d\}$.

Take L_m with maximal index $a_1 \cdots a_d$ in $\mathbf{Z}$ and let $\mathbf{k} = (1/a_1, \ldots, 1/a_d)$.

$\widehat{1_{\mathbf{Z}^d}} = \sum_i \widehat{f_i}$ vanishes at $\mathbf{k}$ but $\widehat{f_m}$ does not, so there exists $j \neq m$ for which $\widehat{f_j}$ does not vanish at $\mathbf{k}$, and our choice of L_m implies $L_j = L_m$. $\diamondsuit$

What about tilings of $\mathbf{Z}^d$ by non-straight sublattices?

In this broader setting the claim can fail. E.g., $\mathbf{Z}^3$ can be written as the disjoint union of four sets, each of which is a translated sublattice of $\mathbf{Z}^3$, no two of which are translates of each other:

$$S_1 = \{(i,j,k) : \ 2|i \text{ and } 2\not| j\}$$

$$S_2 = \{(i,j,k) : \ 2|j \text{ and } 2\not| k\}$$

$$S_3 = \{(i,j,k) : \ 2|k \text{ and } 2\not| i\}$$

$$S_4 = \{(i,j,k) : \ i \equiv j \equiv k \bmod 2\}$$

QUESTION: Can $\mathbf{Z}^2$ be writen as a disjoint union of $n > 1$ translates of sublattices of $\mathbf{Z}^2$ no two of which are translates of each other?

We hope to use elliptic functions and/or theta functions to resolve this question.

QUESTION: If $\mathbf{Z}^d$ $(d \geq 2)$ is written as a disjoint union of $n > 1$ translates of sublattices of $\mathbf{Z}^d$, must two of the lattices be related by rotation?

(Note that for our $\mathbf{Z}^3$ example, the lattices associated with the sets S_1, S_2, S_3 are all related by rotation.)