Why 0.999... is greater than 1.000...

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March 28, 2010

An example of carry-arithmetic with infinite decimals:

$+$	$1.000\ldots$	$0.999\ldots$
$1.000\ldots$	$2.000\ldots$	$1.999\ldots$
$0.999\ldots$	$1.999\ldots$	$1.999\ldots$

Puzzle: Show that if you add two (positive) infinite decimals that don't both end in $000\ldots$, then the sum can't end in $000\ldots$.

Likewise: If you multiply two infinite decimals that don't both end in 000..., then the product can't end in 000....

For instance, multiply .8181... = 9/11 by 1.2222... = 11/9:

$$\begin{array}{cccccccc}
. & 8 & 1 & 8 & 1 & 8 & 1 & \dots \\
. & 1 & 6 & 3 & 6 & 3 & 6 & \dots \\
. & & 1 & 6 & 3 & 6 & 3 & \dots \\
. & & & 1 & 6 & 3 & 6 & \dots \\
. & & & & 1 & 6 & 3 & \dots \\
. & & & & & 1 & 6 & \dots \\
. & & & & & & 1 & \dots \\
. & & & & & & & \dots \\
\hline
. & 9 & 9 & 9 & 9 & 9 & 9 & \dots
\end{array}$$

But wait:

Why does $0.999\ldots$ plus $0.999\ldots$ equal $1.999\ldots$?

We do carries: a "digit" that's ≥ 10 gets reduced by 10, and the digit to its left gets increased by 1.

We can't start our carrying from the rightmost position, since there is no rightmost position.

Q. In what order do we carry?

$$\begin{array}{rcccccccc} & 0 & . & 9 & 9 & 9 & 9 & 9 & \ldots \\ + & 0 & . & 9 & 9 & 9 & 9 & 9 & \ldots \\ \hline \end{array}$$

$$
\begin{array}{cccccccc}
 & 0 & . & 9 & 9 & 9 & 9 & 9 & \ldots \\
+ & 0 & . & 9 & 9 & 9 & 9 & 9 & \ldots \\
\hline
 & 0 & . & 18 & 18 & 18 & 18 & 18 & \ldots
\end{array}
$$

$$
\begin{array}{ccccccccc}
 & 0 & . & 9 & 9 & 9 & 9 & 9 & \dots \\
+ & 0 & . & 9 & 9 & 9 & 9 & 9 & \dots \\
\hline
 & 0 & . & 18 & 18 & 18 & 18 & 18 & \dots
\end{array}
$$

	0	.	9	9	9	9	9	...
+	0	.	9	9	9	9	9	...
	1	.	8	18	18	18	18	...

	0	.	9	9	9	9	9	...
+	0	.	9	9	9	9	9	...
	1	.	8	18	18	18	18	...

	0	.	9	9	9	9	9	...
+	0	.	9	9	9	9	9	...
	1	.	8	19	8	18	18	...

$$
\begin{array}{cccccccccc}
 & 0 & . & 9 & 9 & 9 & 9 & 9 & \dots \\
+ & 0 & . & 9 & 9 & 9 & 9 & 9 & \dots \\
\hline
 & 1 & . & 8 & \color{red}{19} & 8 & 18 & 18 & \dots
\end{array}
$$

	0	.	9	9	9	9	9	...
+	0	.	9	9	9	9	9	...
	1	.	9	9	8	18	18	...

	0	.	9	9	9	9	9	...
+	0	.	9	9	9	9	9	...
	1	.	9	9	8	18	18	...

$$
\begin{array}{cccccccccc}
 & 0 & . & 9 & 9 & 9 & 9 & 9 & \ldots \\
+ & 0 & . & 9 & 9 & 9 & 9 & 9 & \ldots \\
\hline
 & 1 & . & 9 & 9 & 8 & 19 & 8 & \ldots
\end{array}
$$

$$\begin{array}{ccccccccc} & 0 & . & 9 & 9 & 9 & 9 & 9 & \dots \\ + & 0 & . & 9 & 9 & 9 & 9 & 9 & \dots \\ \hline & 1 & . & 9 & 9 & 8 & \color{red}{19} & 8 & \dots \end{array}$$

$$
\begin{array}{cccccccc}
 & 0 & . & 9 & 9 & 9 & 9 & 9 & \ldots \\
+ & 0 & . & 9 & 9 & 9 & 9 & 9 & \ldots \\
\hline
 & 1 & . & 9 & 9 & \color{green}{9} & \color{green}{9} & 8 & \ldots
\end{array}
$$

Q. In what order do we carry?

A. Any order we like!

(as long we don't engage in
"infinite procrastination" anywhere)

The order in which we do the carries doesn't matter; as long as we eventually do a carry in every position, we'll still get 1.999... in the end, in the sense that each and every digit to the right of the decimal point eventually becomes, and stays, a 9.

For other addition problems, the same sort of property holds: the final sequence of digits doesn't depend on the order in which we do the carries, as long as we make sure that once a carry becomes possible in a position, we eventually do a carry in that position.

This kind of independence also holds for a game on graphs called the **chip-firing game** or **sandpile game**.

In one dimension, sandpiles are strings of 0's and 1's (and sometimes, temporarily, larger digits) with an interesting definition of addition: You add in each position, and the carry rule is that a digit that's ≥ 2 gets reduced by 2, and the digit to its left *and* the digit to its right get increased by 1. (1's that get carried off at the far left or the far right of the string disappear.)

We think of a 2 as representing two "chips" or "grains of sand" at a site, which get distributed between the two neighboring sites. We call this operation **firing** the chips or **toppling** the grains of sand.

An example of sandpile arithmetic:
10101 plus 00100 equals 10201
which becomes 11011 after the 2 topples.

Another simple example of sandpile arithmetic:
$11 + 11 = 22 \Rightarrow (1)03 \Rightarrow (1)11(1) = 11.$

Here's a small addition table for the sandpile game:

+	00	11
00	00	11
11	11	11

This should remind you of the situation with numbers that end in 999... and numbers that end in 000....

11 is dominant over 00, just as ending in 999. . . is dominant over ending in 000. . . .

Here's the full sandpile addition table for bit-strings of length 2:

+	00	11	01	10
00	00	11	01	10
11	11	11	01	10
01	01	01	10	11
10	10	10	11	01

Here's the full sandpile addition table for bit-strings of length 2:

+	00	11	01	10
00	00	11	01	10
11	11	11	01	10
01	01	01	10	11
10	10	10	11	01

If we omit the rows and columns labelled 00, the 3-by-3 table that's left is the table for mod-3 arithmetic, where 0, 1, and 2 are represented by 11, 01, and 10.

+	11	01	10
11	11	01	10
01	01	10	11
10	10	11	01

If we omit the rows and columns labelled 00, the 3-by-3 table that's left is the table for mod-3 arithmetic, where 0, 1, and 2 are represented by 11, 01, and 10.

+	0	1	2
0	0	1	2
1	1	2	0
2	2	0	1

In sandpile-arithmetic, we say that 11 and 00 are equivalent, but that 11 is recurrent whereas 00 is not.

+	00	11
00	00	11
11	11	11

Guided by the analogy between chip-firing groups and the semigroup of real numbers, we can recreate the positive reals from their decimal expansions, and we in fact end up learning that 0.999... and 1.000... are equivalent.

+	1.000...	0.999...
1.000...	2.000...	1.999...
0.999...	1.999...	1.999...

When we extend the notion of recurrence to infinite groups and semigroups in the appropriate way, we find that 0.999... is recurrent whereas 1.000... is not.

For more details, see *Carrying On with Infinite Decimals* in the Exchange Book or at http://jamespropp.org/carrying.pdf .

These slides are at http://jamespropp.org/g4g9-slides.pdf.