A Galois Connection in the Social Network

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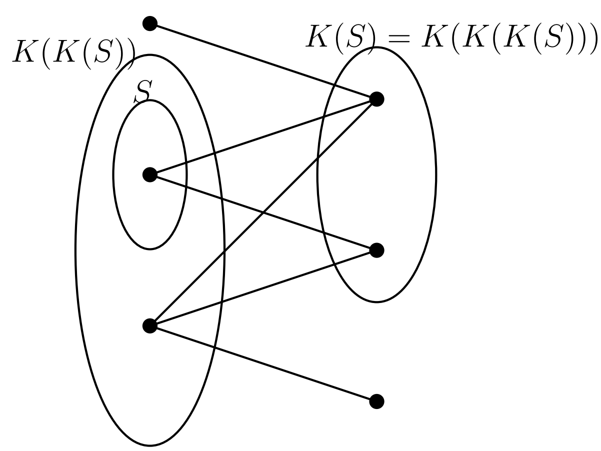
Assume that knowing is a symmetric relation, so that A knows B if and only if B knows A. (This symmetry holds for some sorts of acquaintanceship, such as the "friending" relationship on Facebook.)

Theorem:

The people who know
all the people who know
all the people you know
all are people you know

and the people you know
all are people who know
all the people who know
all the people you know.



A small social network.

Proof: For any set S of people, let $K(S)$ denote the set of people who know *everyone* in S. The accompanying Figure shows a small bipartite example, with dots representing people and edges joining people who know each other. K is **inclusion-reversing**: $S \subseteq S'$ implies $K(S') \subseteq K(S)$.

It is not hard to see that

$$S \subseteq K(K(S)) \tag{1}$$

Applying the inclusion-reversing property to (1) yields

$$K(K(K(S))) \subseteq K(S) \tag{2}$$

On the other hand, replacing S by $K(S)$ in (1) gives

$$K(S) \subseteq K(K(K(S))) \tag{3}$$

The two stanzas of the Theorem are obtained by specializing (2) and (3) to the case $S = \{\text{you}\}$. $\square$

Remark: The mathematical claim and its proof are not original. The operation $K(\cdot)$ is an example of an antitone **Galois connection** from the power set of U to itself, where U is the universe of people. The general notion of a Galois connection can be traced at least as far back as Birkhoff [**1**]; see also the other listed References. An antitone Galois connection is a pair of functions $F : A \to B$ and $G : B \to A$ between two partially-ordered sets A and B, such that for all a in A and b in B, $b \leq_B F(a)$ if and only if $a \leq_A G(b)$. In our case, A and B are both the power set of the universe of people, ordered by inclusion, and F and G are both the map K. To see that we have a Galois connection, note that "$b \leq_B F(a)$" is tantamount to the proposition "everyone in the set b knows everyone in the set a", while "$a \leq_A G(b)$" is tantamount to the equivalent proposition "everyone in the set a knows everyone in the set b". Indeed, a matched pair of asymmetric relations such as "likes" and "is liked by" also give rise to a Galois connection, where $F(a)$ is the set of people who like everyone in the set a and $G(b)$ is the set of people who are liked by everyone in the set b. The proof of the Theorem given above is a specialization of the proof that for any Galois connection, $F \circ G \circ F = F$ and $G \circ F \circ G = G$.

Many Galois connections occur in asymmetric settings, and indeed the term originates from one such example that long predates Birkhoff: given a

Galois extension E of a number field F, the symmetric relation of *knowing* used above corresponds to the asymmetric relations of *fixing* and *being fixed by* (where a group-element σ of $\mathrm{Gal}(E/F)$ fixes a field-element x of E if and only if $\sigma(x) = x$). As part of the proof of the Fundamental Theorem of Galois Theory, one shows that, for any subfield K of E/F, the group-elements that fix all the field-elements that are fixed by all the group-elements that fix K all are group-elements that fix K, and vice versa; these are precisely the automorphisms of E/K. Also, the field-elements that are fixed by all the group-elements that fix K themselves form a field, namely the *Galois closure* of K, which contains K. In the social network, one has an analogous closure operator sending S to the set $K(K(S)) \supseteq S$.

Also, if the topological space X is a path-connected, there is a Galois connection between subgroups of the fundamental group of X and path-connected covering spaces of X. The book [**4**] shows how this idea from algebraic topology can be applied to the study of Fuchsian differential equations.

A consequence of (2) and (3) is the equality $K(K(K(S))) = K(S)$. One virtue of our longer way of stating the result — expressing it as *mutual inclusion* of sets rather than *equality* between sets — is that it gives a hint of the proof. As a bonus, the Theorem as worded above can be sung fluidly (albeit incomprehensibly) to the tune of the jig "The Irish Washerwoman" (`http://www.ireland-information.com/irishmusic/theirishwasherwoman.shtml`).

Thanks to Henry Crapo, David Feldman and Neil Strickland for identifying relevant references, and to the two referees and the editor for helpful suggestions.

References

[**1**] Garrett Birkhoff, Lattice Theory. *Amer. Math. Soc. Coll. Pub.*, Vol. **25**, 1940.

[**2**] Henry Crapo, Ordered Sets: Retracts and Connections, *J. Pure and Appl. Alg.* **23** (1982), 13–28.

[**3**] M. Erne, J. Koslowski, A. Melton, and G.E. Strecker, A Primer on Galois Connections, *Ann. New York Acad. Sci.* **704** (1993), 103–125; also at `http://www.math.ksu.edu/~strecker/primer.ps`.

[**4**] Michio Kuga, Galois' Dream: Group Theory and Differential Equations. Birkhauser, 1992.

[**5**] Øystein Ore, Galois Connexions, *Trans. Amer. Math. Soc.* **55** (1944), 493–513.

[**6**] Peter Smith, The Galois Connection between Syntax and Semantics; `http://www.logicmatters.net/resources/pdfs/Galois.pdf`.

[**7**] Wikipedia article `http://en.wikipedia.org/wiki/Galois_connection`.