Combinatorics and Exact Enumeration in Dimer Models

Lecture #1

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(with helpful comments from Travis Scrimshaw)

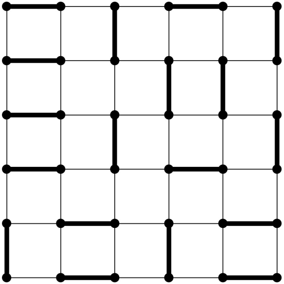
ITS Summer School on Dimers
August 14, 2023

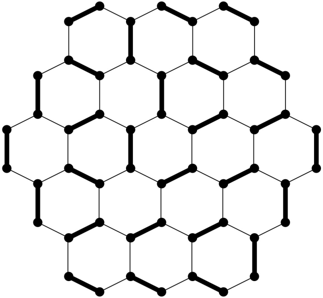
Slides for this talk and the group work assignment are at

http://faculty.uml.edu/jpropp/its1.pdf and

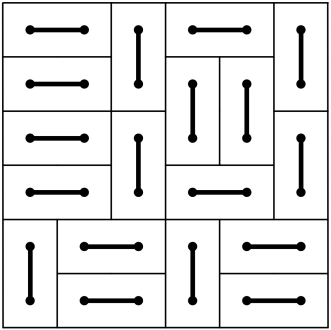
http://faculty.uml.edu/jpropp/its-P1.pdf

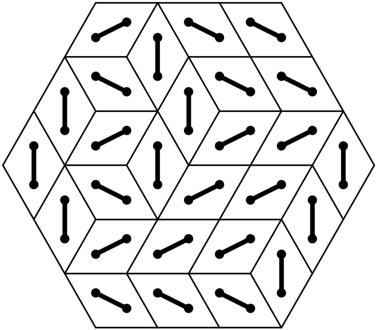
I. Introduction



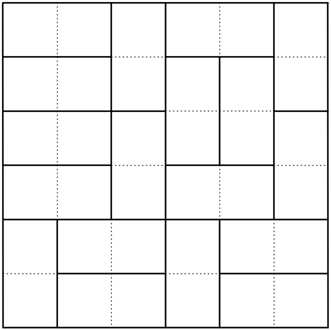


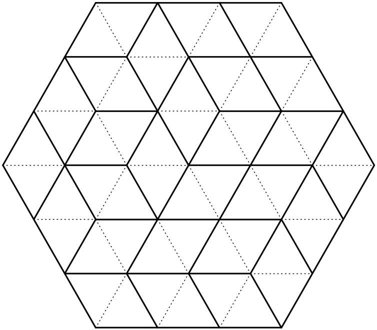
From dimer covers ...





… to tilings





Nomenclature

"Matching" means "perfect matching", aka 1-factor, aka dimer cover.

A weighted graph is a graph with nonnegative real weights assigned to its edges. (Unmarked edges have weight 1.)

The weight of a matching is the product of the weights of its constituent edges. Enumerating the matchings means computing the sum of the weights of all the matchings of G, denoted by $M(G)$.

The union of two squares that share an edge is a domino.

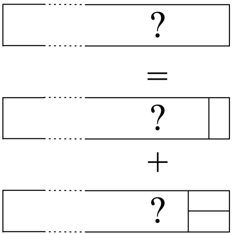
The union of two equilateral triangles that share an edge is a lozenge or rhombus.

The "1-dimensional" theory of tilings

The number of domino tilings of a 2-by-n rectangle (call it T_n) is the coefficient of x^n in the generating function

$$1 + 1x + 2x^2 + 3x^3 + \cdots = \frac{1}{1 - x - x^2}$$

$$T_n = T_{n-1} + T_{n-2}$$



The "1-dimensional" theory of tilings

More generally, for fixed m, the number of domino tilings of an m-by-n rectangle (call it $T(m,n)$) is the coefficient of x^n in a rational generating function.

Idea of proof: Keep track of the number of all ways to tile an m-by-n rectangle with lots of kinds of ragged right edge; set up joint first-order recurrence relations linking all of them.

The “1-dimensional” theory of tilings

Still unsolved (Stanley 1985): When this rational function is expressed in reduced form, is the denominator always of degree $2^{\lfloor (m+1)/2 \rfloor}$?

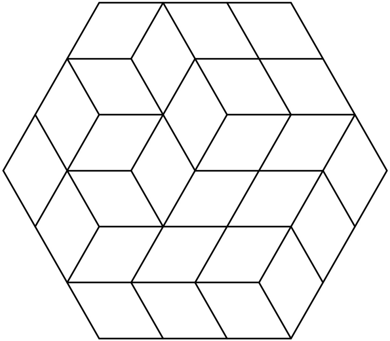
Lagarias proved that it's true when $m+1$ is an odd prime. So for instance, the integer sequence

$$T(100,0), T(100,1), T(100,2), \ldots$$

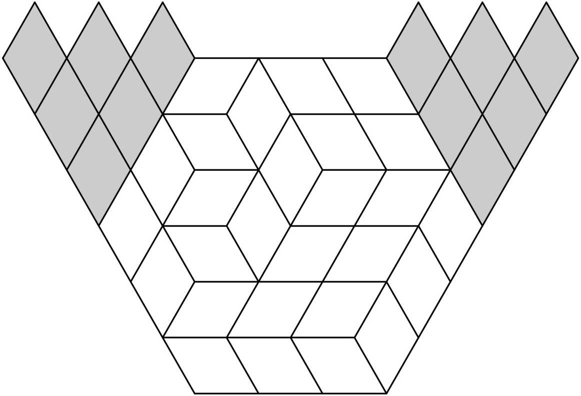
satisfies a linear recurrence of order 1,125,899,906,842,624 (but no smaller).

Not very useful for enumeration.

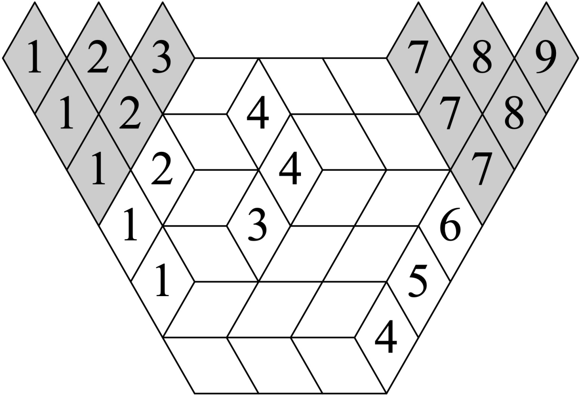
II. Counting lozenge tilings of a regular hexagon



Counting lozenge tilings of a regular hexagon



Counting lozenge tilings of a regular hexagon



Counting lozenge tilings of a regular hexagon

1 2 3 7 8 9

1 2 4 7 8

1 2 4 7

1 3 6

1 5

4

Counting lozenge tilings of a regular hexagon

$$\begin{array}{ccccccccccc} 1 & & 2 & & 3 & & 7 & & 8 & & 9 \\ & 1 & & 2 & & 4 & & 7 & & 8 & \\ & & 1 & & 2 & & 4 & & 7 & & \\ & & & 1 & & 3 & & 6 & & & \\ & & & & 1 & & 5 & & & & \\ & & & & & 4 & & & & & \end{array}$$

Counting lozenge tilings of a regular hexagon

These are semi-strict Gelfand patterns: there is weak increase from left to right along downward-sloping diagonals and strict increase from left to right along upward-sloping diagonals.

There is a bijection between lozenge tilings of the regular hexagon with side-length a and semi-strict Gelfand patterns with top row

$$1 \quad 2 \quad \ldots \quad a \quad 2a+1 \quad 2a+2 \quad \ldots \quad 3a$$

so it suffices to count those.

Let $V(x_1, \ldots, x_n)$ be the number of semi-strict Gelfand patterns with top row $x_1 \ \ldots \ x_n$.

Counting lozenge tilings of a regular hexagon

Claim (Carlitz and Stanley):

$$V(x_1, \ldots, x_n) = \prod_{1 \le i < j \le n} \frac{x_j - x_i}{j - i}$$

Proof: We use induction on n. The claim is trivial for $n = 1$: $V(x_1) = 1$.

Counting lozenge tilings of a regular hexagon

From 1 to 2:

Define the modified summation operator ${}^{L}\sum$ by

$$ {}^{L}\sum_{i=s}^{t} f(i) = \sum_{i=s}^{t-1} f(i) $$

whenever $s < t$. Then

$$ V(x_1, x_2) = {}^{L}\sum_{y_1=x_1}^{x_2} V(y_1) = {}^{L}\sum_{y_1=x_1}^{x_2} 1 = x_2 - x_1 = \frac{x_2 - x_1}{2 - 1} $$

Counting lozenge tilings of a regular hexagon

From 2 to 3:

Extend the definition of ${}^{L}\sum$ by putting

$$ {}^{L}\sum_{i=s}^{s} f(i) = 0 \text{ and} $$

$$ {}^{L}\sum_{i=s}^{t} f(i) = -\,{}^{L}\sum_{i=t}^{s} f(i) \text{ for } s > t $$

so that

$$ {}^{L}\sum_{i=r}^{s} f(i) + {}^{L}\sum_{i=s}^{t} f(i) = {}^{L}\sum_{i=r}^{t} f(i) $$

for all integers r, s, t.

Counting lozenge tilings of a regular hexagon

Define

$$V(x_1, x_2, x_3) = {}^{L}\sum_{y_1=x_1}^{x_2} {}^{L}\sum_{y_2=x_2}^{x_3} V(y_1, y_2)$$

for all integers x_1, x_2, x_3. Now use factor exhaustion:

(1) Show that $V(x_1, x_2, x_3)$, like $\frac{(x_2-x_1)(x_3-x_1)(x_3-x_2)}{(2-1)(3-1)(3-2)}$, is a homogeneous polynomial of degree 3 in x_1, x_2, x_3.

(2) Show that this sum vanishes when $x_1 = x_2$, $x_1 = x_3$, or $x_2 = x_3$.

(3) Show that $V(x_1, x_2, x_3)$ and $\frac{(x_2-x_1)(x_3-x_1)(x_3-x_2)}{(2-1)(3-1)(3-2)}$ have the same coefficient of $x_2 x_3^2$.

For full details, see section 2 of Cohn et al. 1998.

Counting lozenge tilings of a regular hexagon

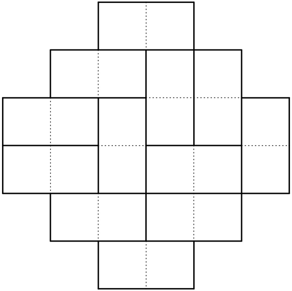
From this approach one can readily derive Macdonald's formula for the number of lozenge tilings of the equiangular hexagon with side-lengths a, b, c, a, b, c:

$$\prod_{i=1}^{a} \prod_{j=1}^{b} \prod_{k=1}^{c} \frac{i+j+k-1}{i+j+k-2}$$

An equivalent formula was given by MacMahon for the number of plane partitions whose 3-dimensional Young diagram fits in an a-by-b-by-c box.

III. Counting domino tilings of an Aztec diamond

The region shown below is an Aztec diamond of order 3.

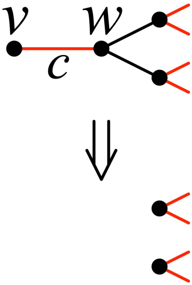


One can enumerate the matchings of the Aztec diamond of order n using the same kind of factor-exhaustion method we used for lozenge tilings of hexagons; see Elkies et al. 1992.

Instead, I'll use the graph-mutation method (sometimes called "urban renewal") that grew out of M. Fisher's work on lattice models in statistical mechanics.

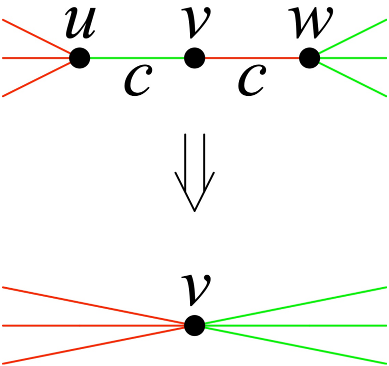
Pruning leaves

Suppose G is a weighted graph with a vertex v of degree 1 joined to its sole neighbor w by an edge of weight c. Let G' be the weighted graph obtained from G by removing v and w and the edge between them. Then $M(G) = c\, M(H)$.



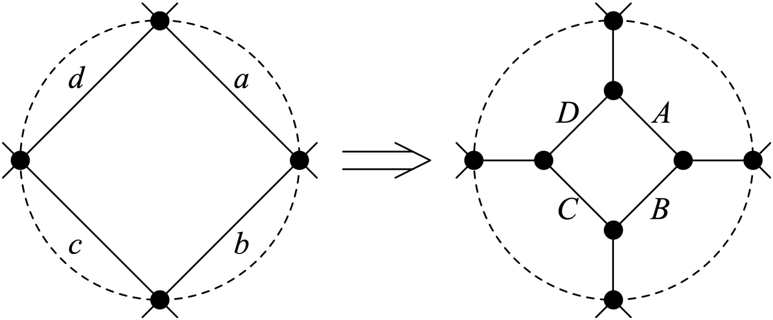
Contracting away degree-2 vertices

Suppose G is a weighted graph with a vertex v of degree 2 joined to neighbors u and w by edges of weight c. Let G' be the weighted graph obtained from G by removing u and w and joining v to every vertex adjacent to u or w (using the same weight). Then $M(G) = c\,M(H)$.

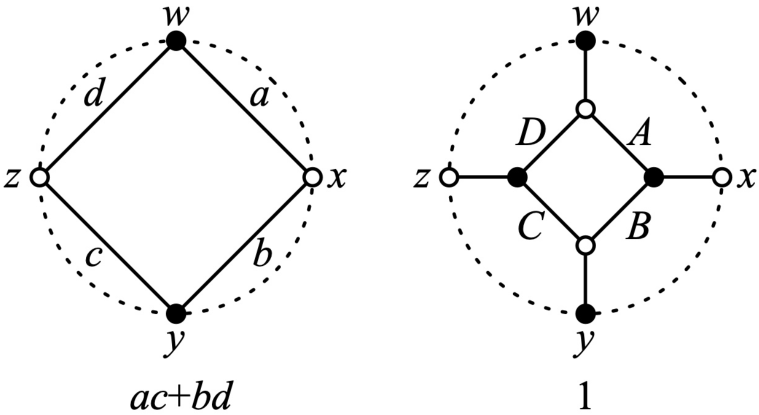


Spider moves

Suppose G is as shown on the left and H is as shown on the right, with $\Delta = ac + bd$, $A = c/\Delta$, $B = d/\Delta$, $C = a/\Delta$, and $D = b/\Delta$, with all external edges and their weights identical. Then $M(G) = \Delta\, M(H)$.

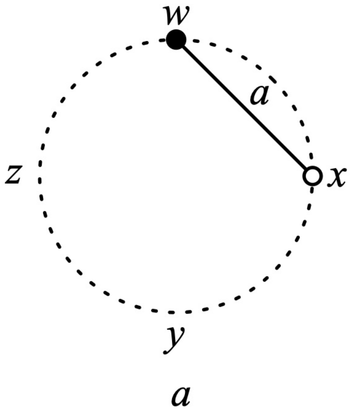


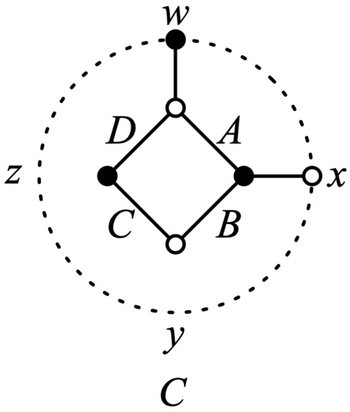
Why the formula holds



$$(ac + bd) = (\Delta)(1)$$

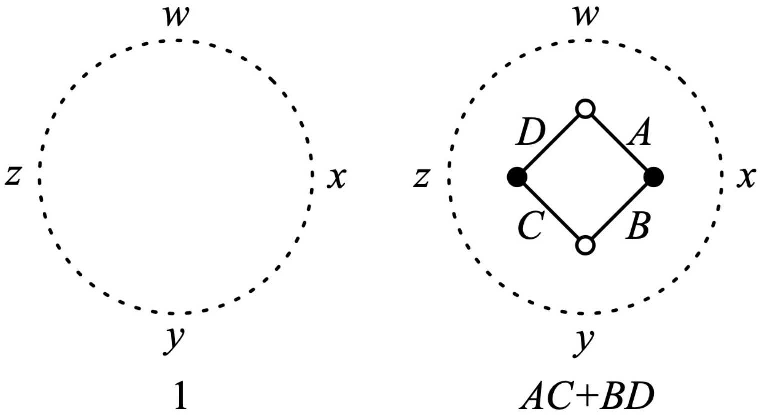
Why the formula holds





$$(a) = (\Delta)(C)$$

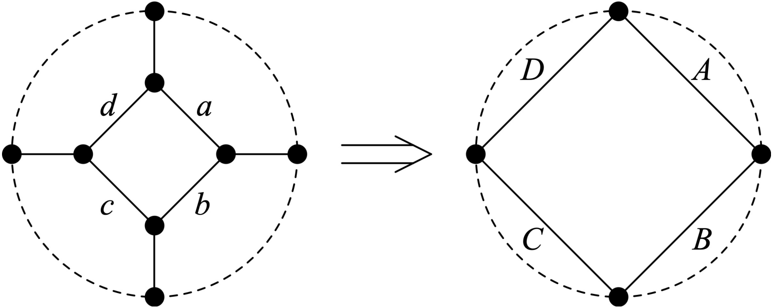
Why the formula holds



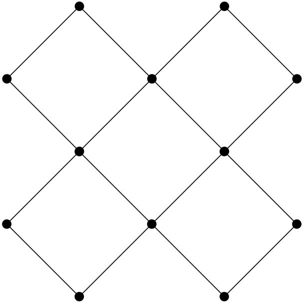
$$(1) = (\Delta)(AC + BD)$$

All moves are reversible

The reverse spider move: Suppose G is as shown on the left and H is as shown on the right, with $\Delta = ac + bd$, $A = c/\Delta$, $B = d/\Delta$, $C = a/\Delta$, and $D = b/\Delta$, with all external edges and their weights identical. Then $M(G) = \Delta\, M(H)$.



Reducing an Aztec diamond of order 2 . . .



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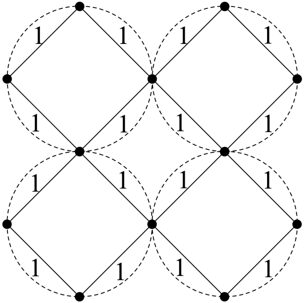
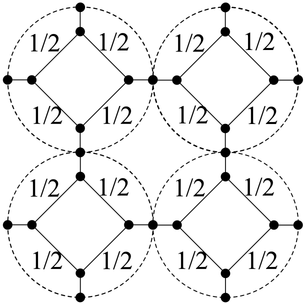


Reducing an Aztec diamond of order 2 . . .



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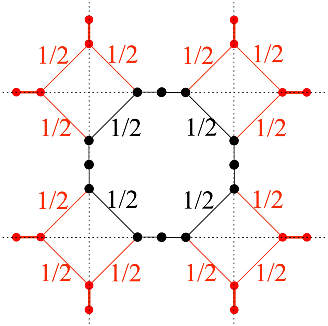
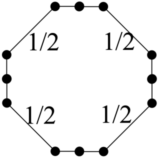
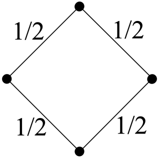


Reducing an Aztec diamond of order 2 . . .



. . . to a weighted Aztec diamond of order 1



A recurrence for matchings of Aztec diamonds

So letting G_n denote the unweighted Aztec diamond graph of order n, we have

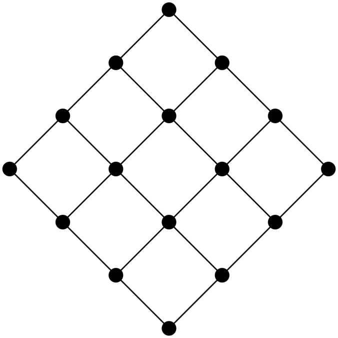
$$\begin{aligned} M(G_2) &= 2^4 \cdot 2^{-2} \cdot M(G_1) \\ &= 2^2 \cdot M(G_1) \\ &= 2^2 \cdot 2^1 \\ &= 2^3 \end{aligned}$$

More generally, $M(G_n) = 2^n\, M(G_{n-1})$, so we obtain the formula of Elkies et al.:

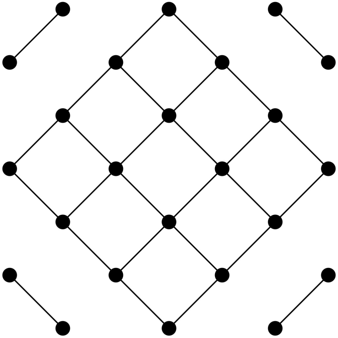
$$\begin{aligned} M(G_n) &= 2^n\, 2^{n-1}\, \cdots\, 2^1 \\ &= 2^{n+(n-1)+\cdots+1} \\ &= 2^{n(n+1)/2} \end{aligned}$$

Not just for Aztec diamonds

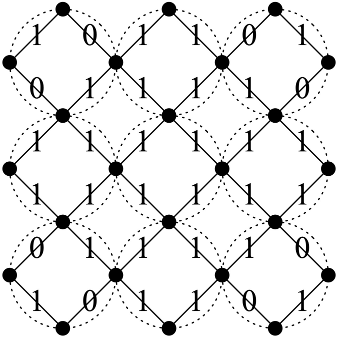
We can use this kind of recurrence on weighted graphs to count matchings of squares.



Not just for Aztec diamonds



Not just for Aztec diamonds



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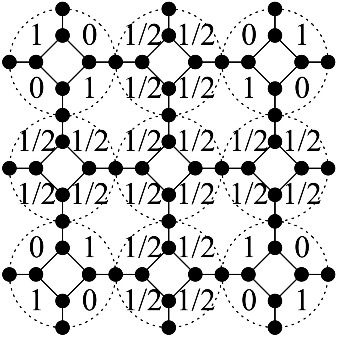

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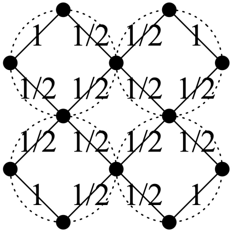

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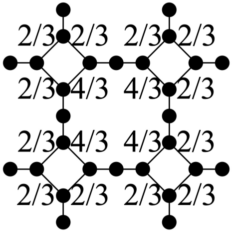

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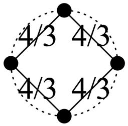


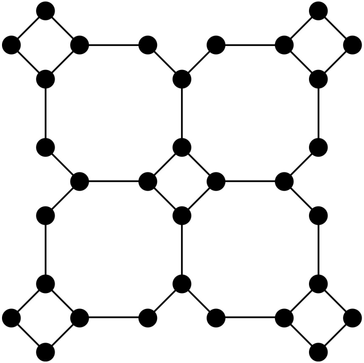
Not just for Aztec diamonds

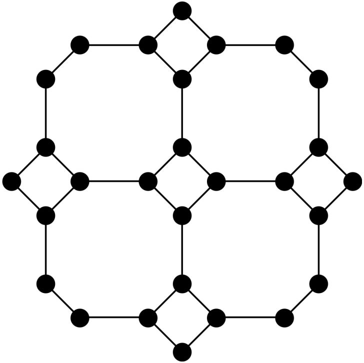
Multiply the Δ-factors:

$$[(1)(2)(1)(2)(2)(2)(1)(2)(1)][(\frac{3}{4})(\frac{3}{4})(\frac{3}{4})(\frac{3}{4})][(\frac{32}{9})] = 36$$

Exercise 1

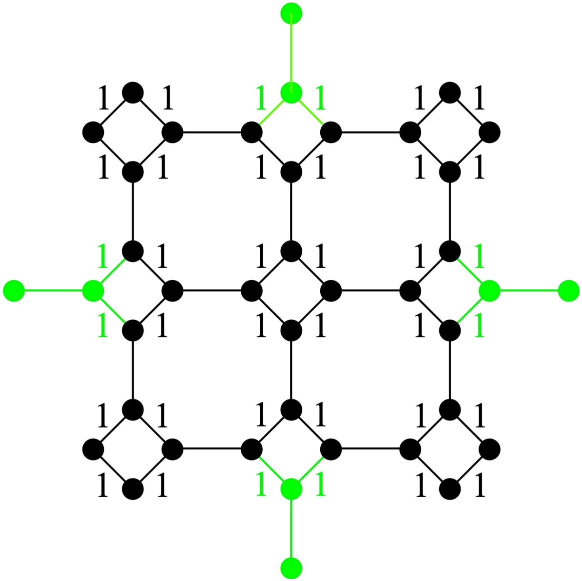
Stick on new vertices and edges of weight 1 and use Δ-factors to count matchings of the two graphs shown below (known as fortress graphs).





Exercise 1

I'll get you started with the first one:



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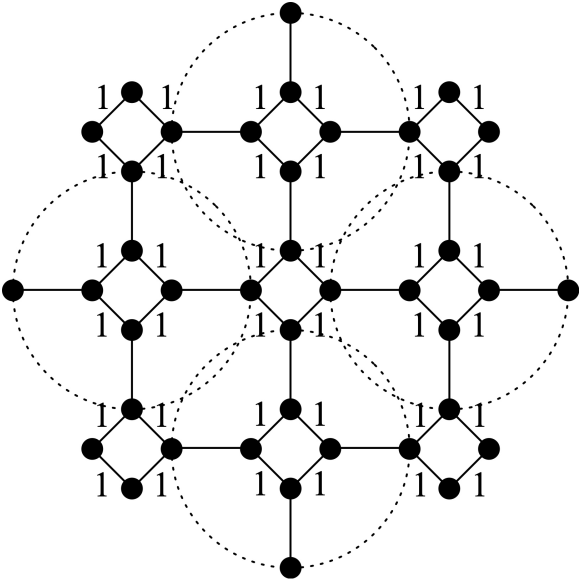


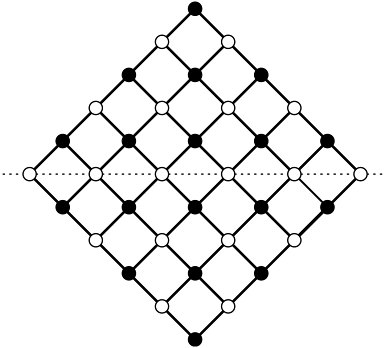
Not just for counting

Graph-mutation can do more than count matchings; it can also calculate the probability that a given edge belongs to a random matching (where the probabilities associated with individual matchings are just their weights, normalized to add up to 1).

In fact, graph-mutation lets you sample from this probability distribution! See Propp 2003. Also see Helfgott's elegant implementation ren.c.

IV. Ciucu's factorization theorem

Suppose G is a weighted bipartite graph embedded in the plane and ℓ is a line in the plane (horizontal for definiteness) such that G with its weight function is symmetric about ℓ. 2-color the vertices white and black.



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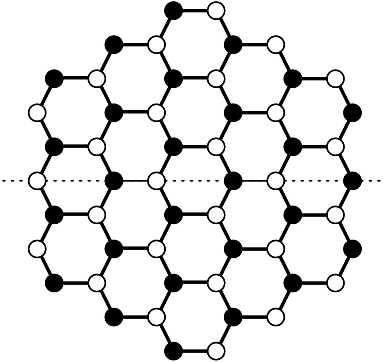
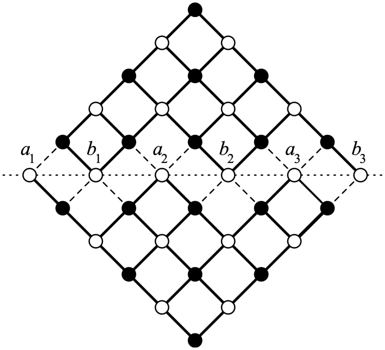


Ciucu's factorization theorem

Suppose that $2k$ vertices of G lie on ℓ and that removing these vertices disconnects G. Label them a_1, b_1, a_2, b_2, ..., a_k, b_k from left to right, and remove edges above all white a_i's and black b_i's and below all black a_i's and white b_i's. Halve the weight of each edge on ℓ.



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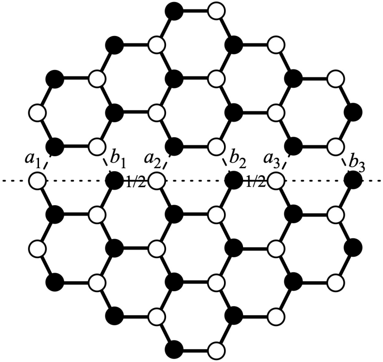


Ciucu's factorization theorem

The new weighted graph is disconnected; let G^+ and G^- be its upper and lower components respectively.

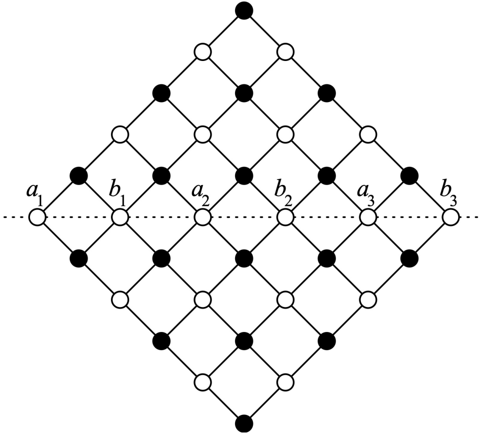
Theorem (Ciucu 1997):

$$M(G) \;=\; 2^k\, M(G^+)\, M(G^-)$$

Example: When G is a $2k$-by-$2k$ square, G^+ and G^- are isomorphic, so $M(G)$ is 2^k times a perfect square.

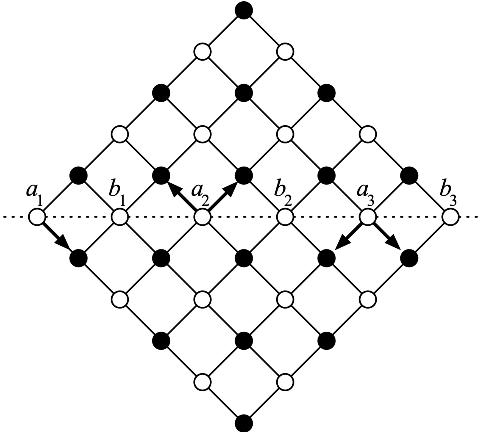
Sketch of proof

I'll confine attention to the special case in which there are only white vertices on ℓ.



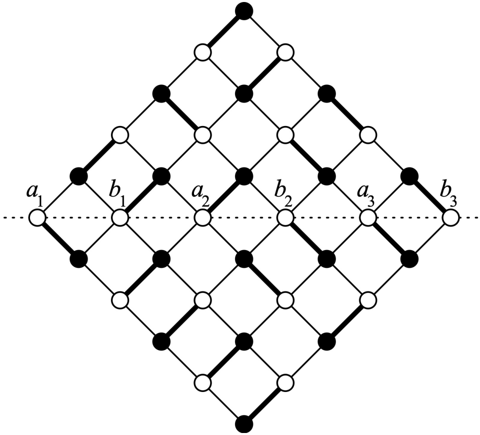
Sketch of proof

We can divide the matchings of G into 2^3 classes according to whether a_1, a_2, a_3 match up or down.



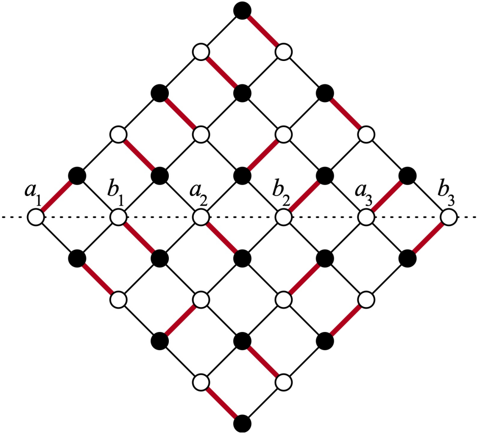
Sketch of proof

I'll show you a bijection that turns a down-up-down matching into a down-down-down matching.



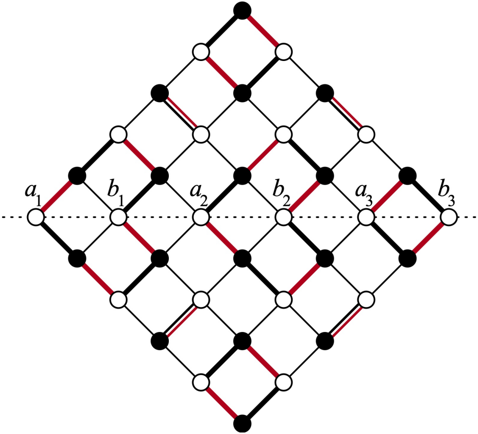
Sketch of proof

Consider the matching (shown in red) obtained by reflecting the down-up-down matching (shown in black before) across ℓ.



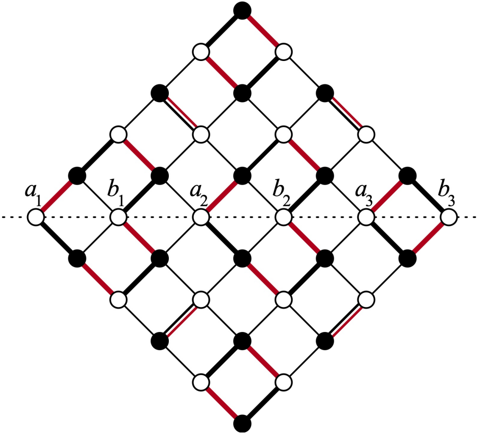
Sketch of proof

When we superimpose the two matchings we get a 2-factor of G with a cycle through a_2.



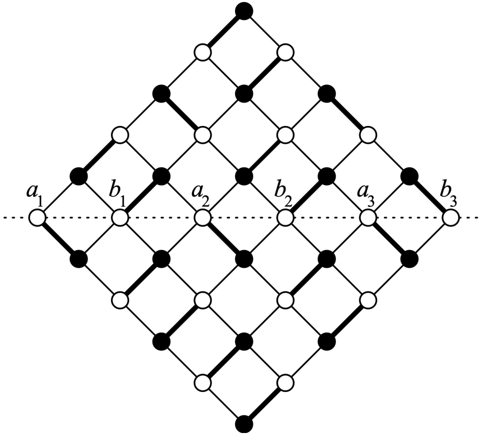
Sketch of proof

Replace the black edges in that cycle by red edges and vice versa.



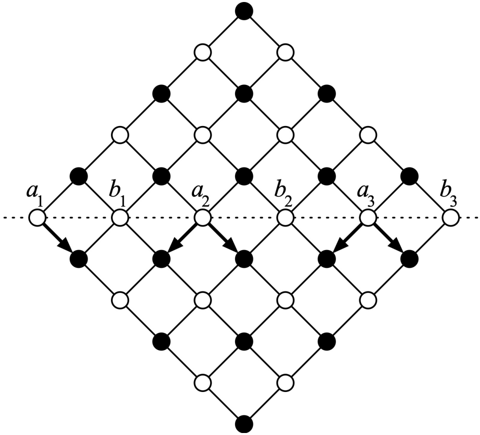
Sketch of proof

The edges that are now colored black form a different matching of G ...



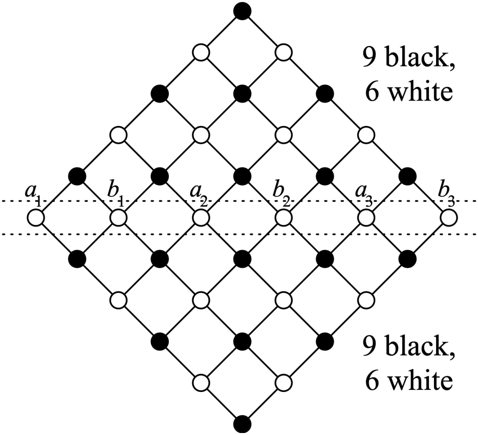
Sketch of proof

. . . and it's of type down-down-down. This construction shows more generally that all eight classes are equinumerous.



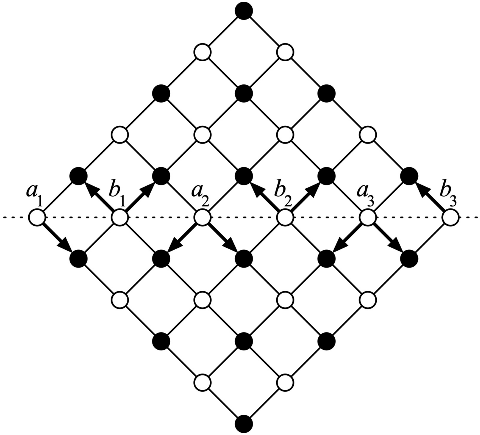
Sketch of proof

Moreover, in the down-down-down class, all the *b*-vertices must match upward!



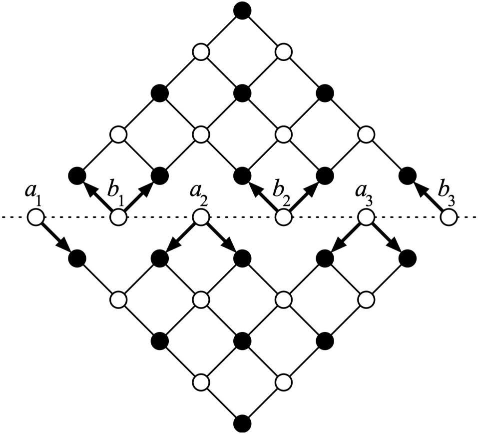
Sketch of proof

So, the number of matchings of G equals 8 times the number of matchings in which a's match down and b's match up.



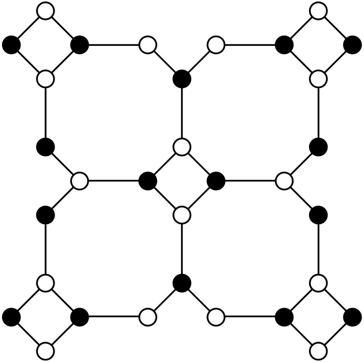
Sketch of proof

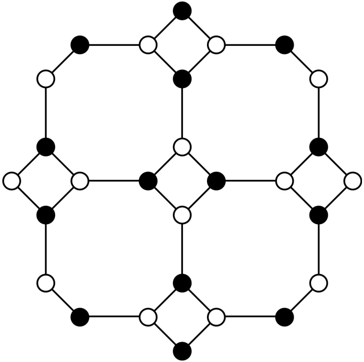
But removing the now-forbidden edges gives a disconnected graph: a copy of G^+ and a copy of G^-.



Exercise 2

Use Ciucu factorization to count the matchings of the graphs from homework problem 1.





References with links

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...and these slides and homework 1.