Self-organizing structures in rotor-router blobs

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Slides for this talk are on-line at
http://jamespropp.org/jwm11.pdf

Acknowledgments

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Warning

In this talk, I ignore factors of π.

E.g., "The area of a disk of radius r is r^2."

How to grow a random blob

Internal Diffusion-Limited Aggregation (Meakin-Deutch, Diaconis-Fulton):
Start a walker at the origin.
As long as the walker is in the current blob, let the walker take random steps.
When the walker reaches a site not in the current blob, add the site to the blob.
Repeat!

Theorem (Lawler-Bramson-Griffeath): The blob of size n (rescaled by $\sqrt{n}$) almost surely converges to the disk of radius 1 as n goes to infinity.

How to grow a non-random blob

Rotor-Router Aggregation (Propp):
Start a walker at the origin.
As long as the walker is in the current blob, let the walker take non-random steps, such that for each vertex v, the sequence of directions followed by the walker immediately after its visits to v is the period-4 sequence

North, West, South, East, North, West, South, East, ...

When the walker reaches a site not in the current blob, add the site to the blob.
Repeat!

Theorem (Levine-Peres): The blob of size n (rescaled by $\sqrt{n}$) converges to the disk of radius 1 as n goes to infinity.

Rotors

It's helpful to imagine that each site comes equipped with a *rotor* that tells the walker where to go.
Specifically, when the walker arrives at a site, the rotor at that site advances (counterclockwise) and the walker moves in the direction that the rotor now points in.
If we represent the four rotor-settings by four colors, the color-map of the n-site aggregate displays fascinating patterns that, from a rigorous perspective, we know virtually nothing about. See

http://www.mathpuzzle.com/29Jun2003.html

Note for instance the presence of mesoscopic monochromatic patches (small compared to the blob, but big compared to the grid-scale); a mystery!

Random vs. non-random: Campaign slogans

#1: "Rotor-Router Aggregation is derandomized Internal DLA"

#2: "Internal DLA is randomized Rotor-Router Aggregation"

One reason I resist slogan #2 is that Rotor-Router Aggregation isn't as symmetrical as Internal DLA.
If say we used the period-4 sequence East, North, West, South, ... instead of the period-4 sequence North, West, South, East, ..., we'd get a different picture for the four-colored blob of size n.

But not as different as you might think!

Conjecture: The two aforementioned pictures differ at $o(n)$ sites as $n \to \infty$.

Symmetry

If this conjecture is true, it follows that large rotor-router blobs have approximate 4-rotational symmetry: if you rotate the blob and permute the colors accordingly, all but $o(n)$ of the sites are the same color as before.

We see this approximate symmetry in simulations.

Indeed, we see an approximate 8-fold rotational symmetry if we look at the locations of the monochromatic patches (look again at `http://www.mathpuzzle.com/29Jun2003.html`).

Where the monochromatic patches are

It's helpful (here and elsewhere) to coordinatize the blob of size n using complex numbers of norm ≤ 1: that is, we rescale the picture by $\sqrt{n}$, so that it becomes a disk of radius 1, and then sit this disk in the complex plane, centered at 0.

Conjecture (Cook, Hoey): The monochromatic patches are at those locations z for which $1/z^2$ is in $1 + 2\mathbf{Z} + 2i\mathbf{Z}$.

E.g., taking z in $\{1, i, -1, -i\}$ so that $1/z^2$ is ± 1, we predict patches at the North, East, South, and West "Poles" of the blob.

The (slightly twisted) 8-fold symmetry in the disk reflects the 4-fold symmetry of the lattice $\mathbf{Z} + i\mathbf{Z}$ (the 4 gets doubled because $z \mapsto 1/z^2$ doubles angles).

For dramatic visual evidence of this conjecture see
http://math.mit.edu/~levine/gallery/invertedrotor1m15x.png

Patches on the move

Michal Falenski's Java applet

http://rotor-router.mpi-inf.mpg.de/applet/

shows how these monochromatic patches move over time.

At the boundary of the disk, we see dart-shaped patches that move outward and disappear from the disk.

In the interior, we see darts that come together from opposite directions, annihilate, and then are recreated, heading outward in the two perpendicular directions.

Stabilization

To see what's unchanging, look at the blob for special values of n, e.g., those values of n such that a new row of length 1 gets created at the top of the blob of size $n+1$:

`http://jamespropp.org/testt.mov`

Unexpected stabilization

David Einstein has observed that we also get stabilization if, for those special values of n, we look at some other (but not all!) locations z in the rescaled size-n blob. Not surprisingly, the pictures stabilize for all z with $z^4 = 1$, but quite curiously, the simplest z with $|z| < 1$ for which this stabilization seems to occur is $z = \frac{1+2i}{5}$, with $1/z^2 = -3 - 4i$.

http://jamespropp.org/pyth.mp4

Is it coincidental that $|-3-4i|$ is an integer?

Where other interesting patches are

If we take $z = \frac{1+i}{\sqrt{2}}$, $1/z^2$ is i, which is not in $1 + 2\mathbf{Z} + 2i\mathbf{Z}$.
But we see something interesting in the North-East corner of the blob: mesoscopic patches in which the colors alternate in checkerboard fashion.

More generally, we predict that if $1/z^2$ is $a + ib$ with $a, b \in \mathbf{Q}$, there will be a mesoscopic patch at z in which the coloring is spatially periodic, with period depending on the denominators of a and b. (We observe this to be true when the denominators are small.)

Join the expotition

Tobias Friedrich, using the methods he and Lionel Levine developed (discussed in his talk), has generated some really big rotor-router blobs and put them on the web, along with a Google-maps zooming interface:

http://rotor-router.mpi-inf.mpg.de/

See e.g.

http://jamespropp.org/zoom1.png
http://jamespropp.org/zoom2.png
http://jamespropp.org/zoom3.png
http://jamespropp.org/zoom4.png

showing successive zooms on the North Pole.
Anyone can jump into this gigapixel image and look for patterns. E.g.,

http://jamespropp.org/illusion.pdf

Using a different kind of rotor

Even more interesting structures appear if we use the period-4 sequence

East, West, North, South, East, West, North, South, ...

instead of the period-4 sequence

East, North, West, South, East, North, West, South, ...

See

http://rotor-router.mpi-inf.mpg.de/1Bio/?rotorseq=2

(also on display in the Mathematical Art Exhibition).

Rick Kenyon noticed that the ghostly necklaces of nearly round beads seen in this picture match up quite well with the picture obtained by conformally mapping a simple circle-packing in the plane by $z \mapsto \pm 1/\sqrt{z}$:

http://jamespropp.org/RRcircles2.pdf

Frustrations of mathematical pointillism

But what mathematical structure corresponds to what our eye and brain see?
Eyes are good at detecting edges.
It seems that the kind of edge our eye detects in one part of the picture is qualitatively different from the kind of edge our eye detects in another!

http://jamespropp.org/curve.png

Moreover, when we zoom in, we tend to lose sight of what we are trying to understand!
To turn sense-impressions into conjectures, I'd need a better understanding of what ImageMagick does and/or what the human visual system does.

High-discrepancy rotors

If one uses non-periodic sequences like

East, North, North, West, West, West,
South, South, South, South, ...

one gets non-round blobs like the two shown in

http://jamespropp.org/TF-A.gif

and

http://jamespropp.org/TF-B.gif

(Contrast this behavior with Friedrich's simulations of random low-discrepancy rotors.)

Derandomized deposition

Rotor-router aggregation was designed to be a deterministic analogue of internal DLA.

Tobias Friedrich has created a simulation of a deterministic deposition model that uses rotor-routers to derandomize directed random walk and uses the walk to deposit particles on a 1-dimensional substrate. Compare

http://jamespropp.org/snow_fullyrnd.mpg

with

http://jamespropp.org/snow.mpg