Bringing tiling theory to new heights and vice versa:

a talk in honor of Rick Kenyon,
a mathematician with no fear of heights

Jim Propp, UMass Lowell
(joint with Colin Defant, Leigh Foster, Rupert Li,
Cris Moore, Hanna Mularczyk, and Benjamin Young)

June 20, 2025

Slides available at http://faculty.uml.edu/jpropp/kenyon.pdf;
video available soon.

A non-paper by Rick

The arXiv preprint by Random trimer tilings by Ghosh and Dhar cites the following paper:

> [20] R. Kenyon, Ann. Prob. **28**, 759 (2000). Cylindrical boundary conditions for the trimer problem correspond to periodic boundary conditions for the height field only if the width m is divisible by 3.

There is no paper by Rick with that title.

He wrote an "Ann. Prob. **28**, 759 (2000)" paper, but it's about conformal invariance of the dimer model.

Rick tells me that no paper of his proves that result about trimers, but that it can be derived from the results in the 1992 paper Tiling a polygon with rectangles by Kenyon and Kenyon.

Kenyon and Kenyon, 1992

Theorem 3 *Any two tilings of P by $m \times 1$ and $1 \times n$ tiles are obtained from one another by "rotations" of the form illustrated figure 14. That is, m vertical $1 \times n$ tiles can be replaced by n horizontal $m \times 1$ tiles and conversely.*

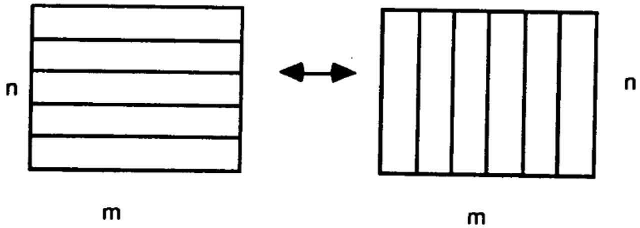
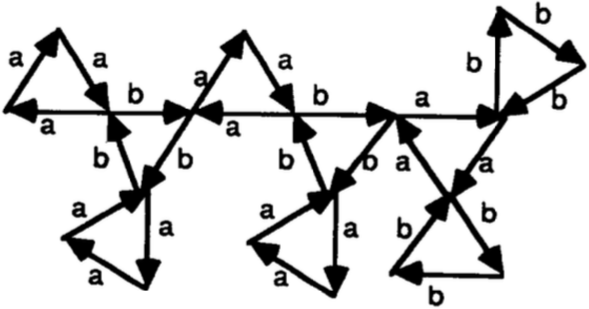


Figure 14: rotations for transforming $m \times 1$, $1 \times n$ tilings

Kenyon and Kenyon, 1992

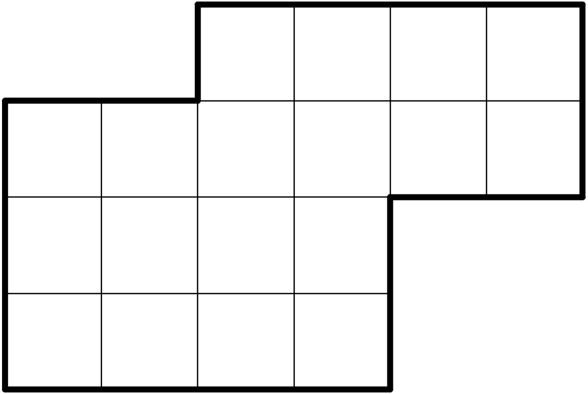
To generalize Thurston's approach, the main problem is finding the correct definition of a height function.

(referring to Conway's Tiling Groups by Thurston (1990))



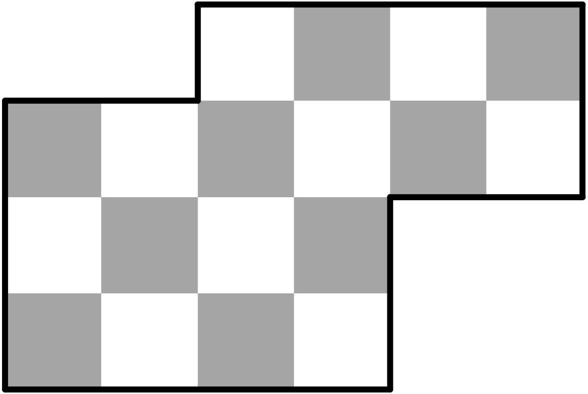
Thurston, 1990

Thurston gives a necessary (not sufficient) criterion for determining when a plane region R can be tiled by dominos.



Thurston, 1990

We two-color the grid of squares.



Thurston, 1990

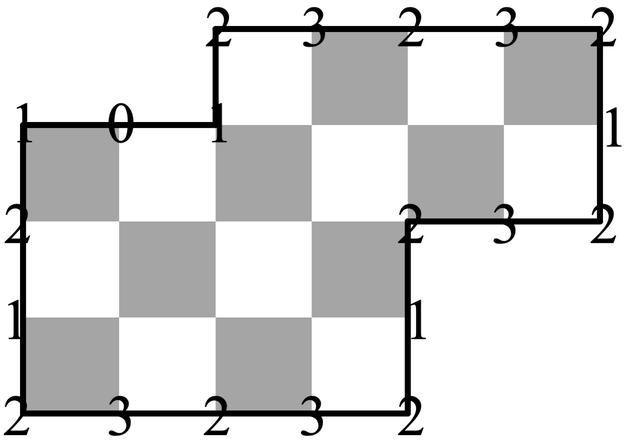
We assign an integer value $h(v)$ to some vertex v on the boundary π of R (the choice of v and the choice of $h(v)$ do not matter) and we go around the boundary of R, extending $h(\dots)$ to each new vertex we encounter:

> The rule is simple: one can start with 0 at some arbitrary vertex. Along any edge of π which has a black square to its left, the height increases by 1. Along any edge with a white square to its left, the height decreases by 1. A necessary condition that R can be filled with dominoes is that the height after traversing once around the curve is 0.

We call $h(\cdot)$ a **height-function**.

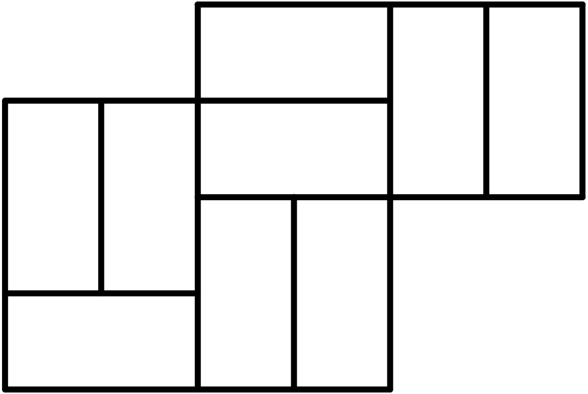
Thurston, 1990

Applying this to our example:



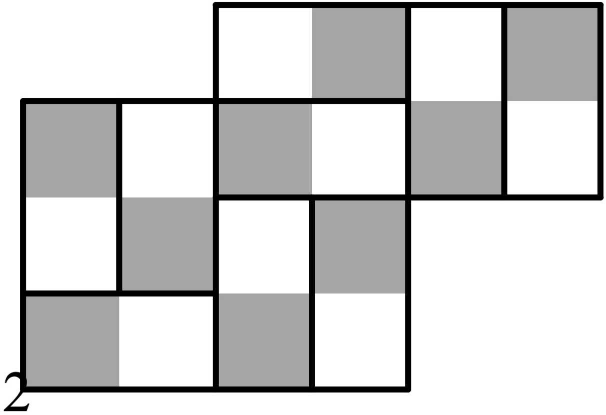
Thurston, 1990

To see why this works, suppose there were a tiling of R:



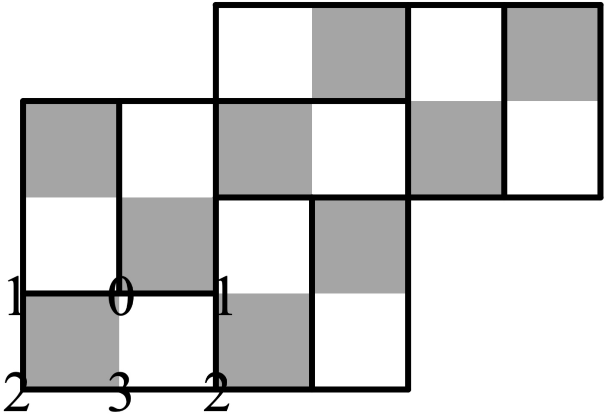
Thurston, 1990

Extend $h(\cdot)$ to R one tile at a time:



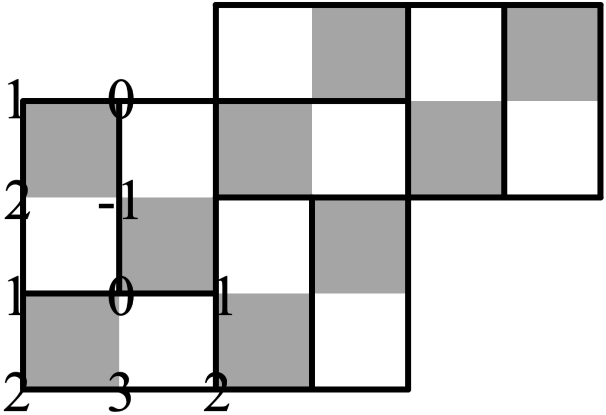
Thurston, 1990

Extend $h(\cdot)$ to R one tile at a time:



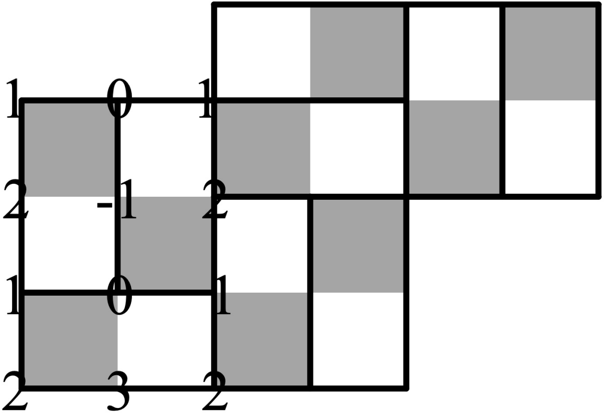
Thurston, 1990

Extend $h(\cdot)$ to R one tile at a time:



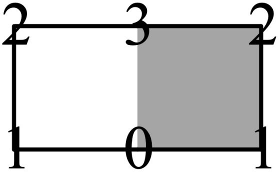
Thurston, 1990

Extend $h(\cdot)$ to R one tile at a time:



Thurston, 1990

The height function associated with a domino tiling is locally consistent because each tile is color-balanced (going around a tile causes the height to change by $3 - 3 = 0$) and it's globally consistent because of local consistency combined with the hypothesis that the region is simply-connected.



Thurston, 1990

In this way, the domino tilings of R correspond to the height-functions on R.

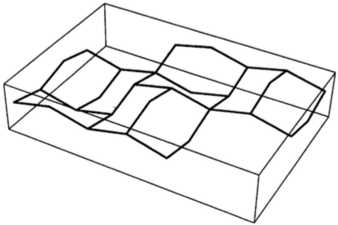
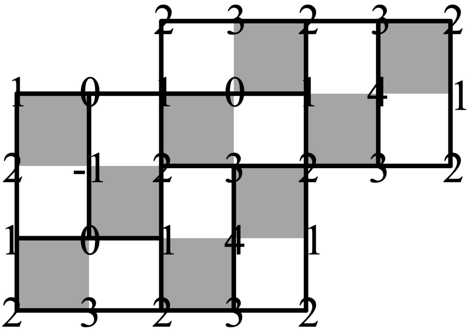


FIG. 4.3. Domino tiling. A tiling by 9 dominoes, lifted to the graph of the domino group.

The map from tilings to height-functions is many-to-one, but can be made unique if we fix the coloring and fix the height of a designated boundary vertex.

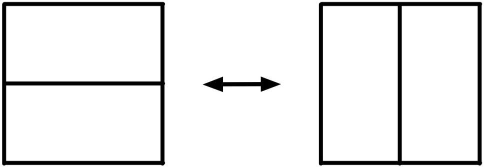
Thurston, 1990

The height function retains all the information in the tiling: two vertices p, q that are adjacent in the underlying cell-complex of squares are joined by an edge in the tiling if and only if the corresponding height function h has the property that $|h(p) - h(q)|$ is 1.



Converting one tiling into another

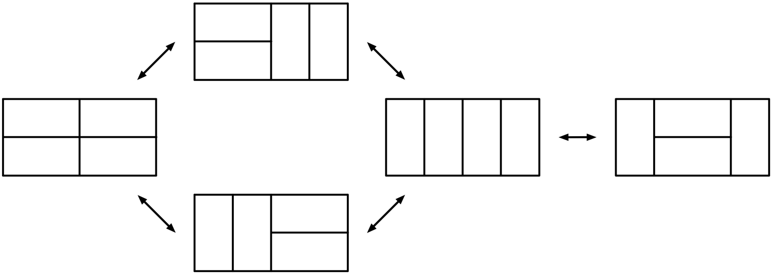
Thurston was aware that his height-functions allow one to prove that for any two domino-tilings t_1 and t_2 of a simply-connected plane region, t_1 can be converted into t_2 through a succession of "flips", each of which retiles a 2-by-2 square (replacing horizontal dominoes by vertical dominoes or vice versa).



The first published statement of this result was in the 1995 paper Spaces of Domino Tilings by Saldanha, Tomei, Casarin, and Romualdo.

Saldanha et al., 1995

Two tilings are *adjacent* in T if we move from one to the other by a flip. Turn T into a graph by joining adjacent tilings by edges and define connected components of T and distance between tilings in the usual way.



T is the "space" (i.e., graph) of tilings whose vertices correspond to tilings and whose edges correspond to flips.

Saldanha et al., 1995

Our techniques provide us with a fair understanding of the combinatorial, topological, and metric structure of T

. . .

we describe in Theorem 3.2 a simple formula for the distance between tilings and a characterization of shortest routes between points.

. . .

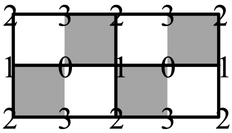
$$d(t_1, t_2) = \frac{1}{4} \sum_{p \in A^*} |\theta_1(p) - \theta_2(p)|.$$

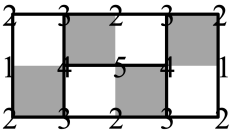
Here t_1 and t_2 are tilings of a simply-connected region, $d(t_1, t_2)$ is the flip-distance between them, A^* is the set of vertices of the square cells that underlie both tilings, and θ_1 and θ_2 are height functions associated with t_1 and t_2.

Saldanha et al., 1995

Recall that height goes up by 1 (resp. down by 1) when you travel along a tile-edge with a shaded (resp. unshaded) square to your left.

Check the formula for $d(t_1, t_2)$:





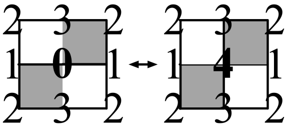
$$\frac{1}{4}\big(|0-4| \;+\; |1-5| \;+\; |0-4|\big) \;=\; 3$$

Saldanha et al., 1995

Notice that if t_1 and t_2 are two domino tilings that are related by a single flip,

$$\frac{1}{4}\sum_{p\in A^*}|\theta_1(p)-\theta_2(p)| = 1$$

because $\theta_1(p)-\theta_2(p) = \pm 4$ if p is the center of the 2-by-2 square that has been flipped and $\theta_1(p)-\theta_2(p) = 0$ otherwise.



Saldanha et al., 1995

This equation implies, by way of the triangle inequality, that for general t_1 and t_2,

$$d(t_1, t_2) \geq \frac{1}{4} \sum_{p \in A^*} |\theta_1(p) - \theta_2(p)|$$

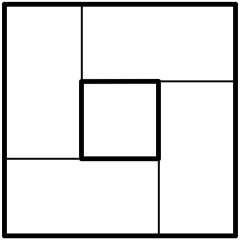
The real work is showing that equality holds.

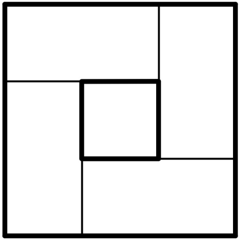
Once that work is done, we obtain an interpretation of the individual terms of the sum: $\frac{1}{4}|\theta_1(p) - \theta_2(p)|$ equals the number of flips that get performed on the 2-by-2 square centered at p along every geodesic in T from t_1 to t_2.

Saldanha et al., 1995

Saldanha et al. consider extensions to other sorts of regions, but for now, let's just note that if R is not a simply-connected subset of the plane, the height-function may not even be well-defined.

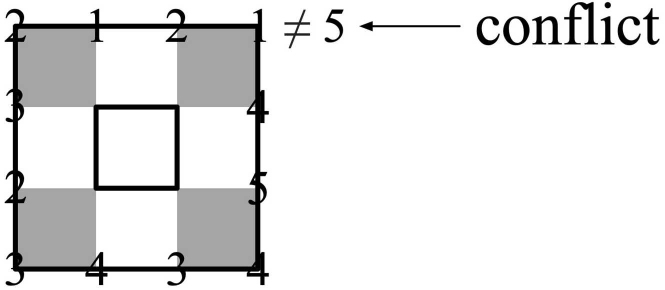
E.g., consider a 3-by-3 square with a 1-by-1 hole in the middle. The region can be tiled (and has exactly two tilings, not related by a flip of the kind we're considering).





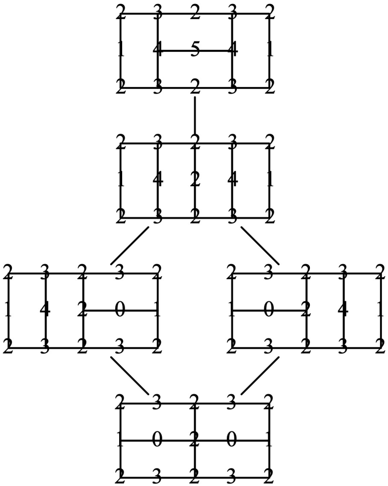
Saldanha et al., 1995

Any putative height-function would have to increase by 4 along the outer boundary, which is impossible.



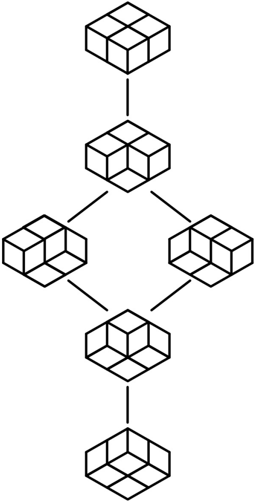
Propp, (1993) 2025

In the 1990s, I showed that the natural partial order on height-functions makes the space of domino-tilings of a simply-connected region a distributive lattice.



Propp, (1993) 2025

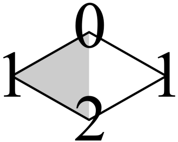
Many other combinatorial models have the same property, such as tilings of hexagons by lozenges (two equilateral triangles that share an edge).



Thurston, 1990

Shade all left-pointing triangles. Then height goes up (resp. down) by 1 when you travel along a tile-edge with a shaded (resp. unshaded) triangle to your left.

The height function is globally consistent and determines the tiling: an edge with endpoints p, q is a tile-edge if and only if $|\theta(p) - \theta(q)| = 1$.

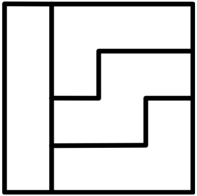


Saldanha showed that for lozenge tilings of simply-connected regions,

$$d(t_1, t_2) = \frac{1}{3} \sum_p |\theta_1(p) - \theta_2(p)|$$

Ribbon tilings

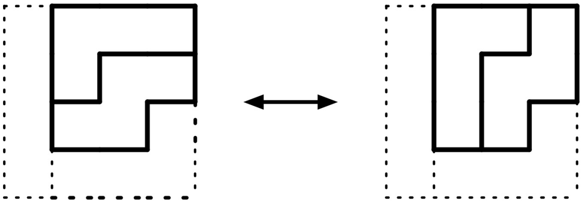
Igor Pak and others extended the Conway-Lagarias-Thurston approach to the study of ribbon tilings, e.g., 4-ribbon tilings:



See On tilings by ribbon tetrominoes by Muchnik and Pak (1999), Ribbon tile invariants by Pak (2000), and Ribbon tile invariants from the signed area by Moore and Pak (2002).

Ribbon tilings

In his 2002 article Ribbon tilings and multidimensional height functions , Scott Sheffield used height functions to resolve a conjecture of Pak's, showing that for any two k-ribbon tilings t_1, t_2 of a simply-connected region, t_2 can be obtained from t_1 via a sequence of 2-flips like this:



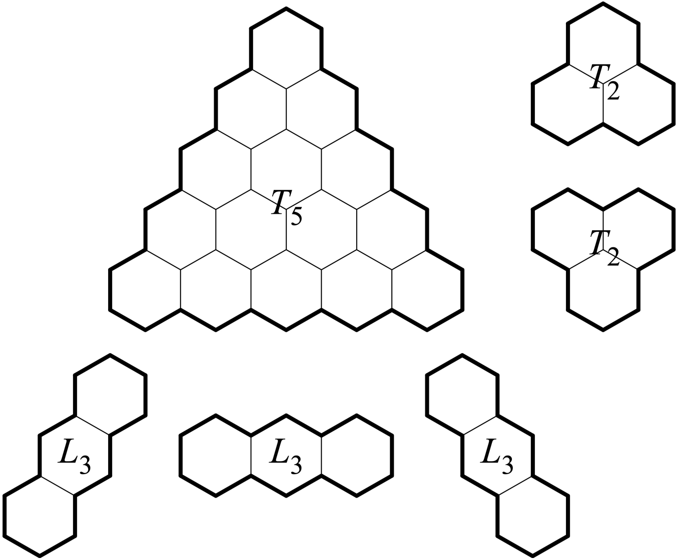
Conway and Lagarias, 1990

All of this work was directly or indirectly inspired by the 1990 article of Tiling with polyominoes and combinatorial group theory by Conway and Lagarias. As Saldanha writes:

> Conway and Lagarias [1] studied the problem of tiling a subset of $\mathbb{R}^2$ with a given set of tiles, by group-theoretical techniques. Thurston [10] adapted these techniques to study domino tilings, producing a necessary and sufficient condition for a simply connected region of the plane to be tileable by dominoes.

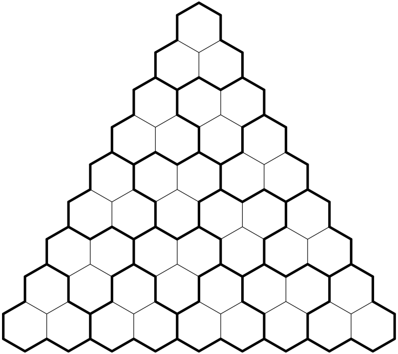
Conway and Lagarias' work concerns tiles made of three hexagons (unlike dominoes which are made of two squares).

Conway and Lagarias, 1990



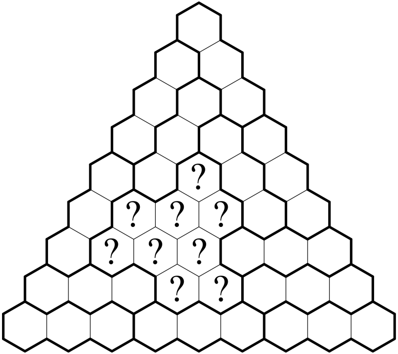
Conway and Lagarias

Theorem (C & L):
(1) T_n can be tied by T_2's precisely when n is 0, 2, 9, or 11 (mod 12).



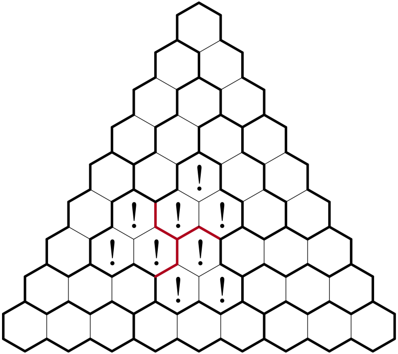
Conway and Lagarias

Theorem (C & L):
(2) T_n can be tied by L_3's precisely when n is 0 (i.e., never, outside of the trivial case).



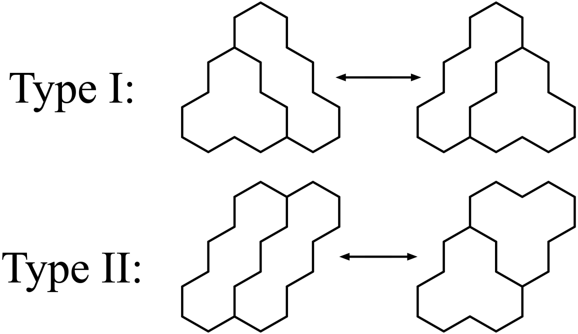
Conway and Lagarias

But what if we allow both kinds of tiles?
Then as long n is congruent to 0 or 2 mod 3 (so that the number of hexagons is a multiple of 3), T_n can be tiled by a mix of T_2 tiles and L_3 tiles.



An old (?) conjecture

Conjecture: For any two T_2-and-L_3 tilings t_1 and t_2 of a simply-connected region, t_1 can be converted into t_2 through a succession of moves of the following two kinds:



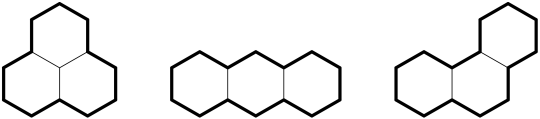
An old (?) conjecture

I'm not sure who first conjectured this, but my best guess is that it was me, circa 2000 (in emails).

In any case, Colin Defant, Leigh Foster, Rupert Li, Hanna Mularczyk, Cris Moore, Benjamin Young and I are working toward a proof.

New terminology

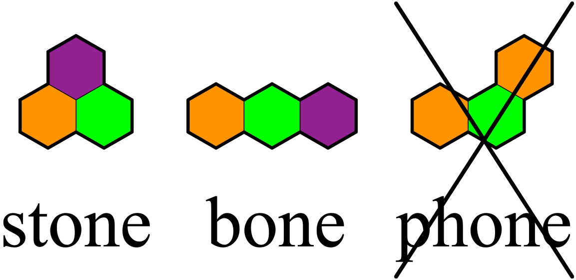
The T_2 and L_3 tiles are two of the three kinds of "trihexes". I rebranded them as the stone, the bone, and the phone.



stone bone phone

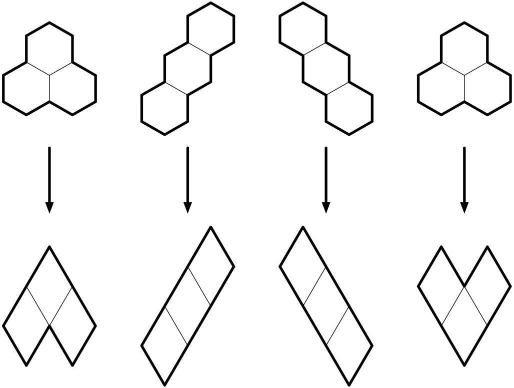
New terminology

One reason to expect the stone and bone tiles to be more tractable than the phone tile is that, if one three-colors the grid of hexagons (Orange, Green, Purple) so that adjacent hexagons are different color, then a stone or bone contains one hexagon of each color while a phone does not. So let's follow Conway and Lagarias and ban phones.



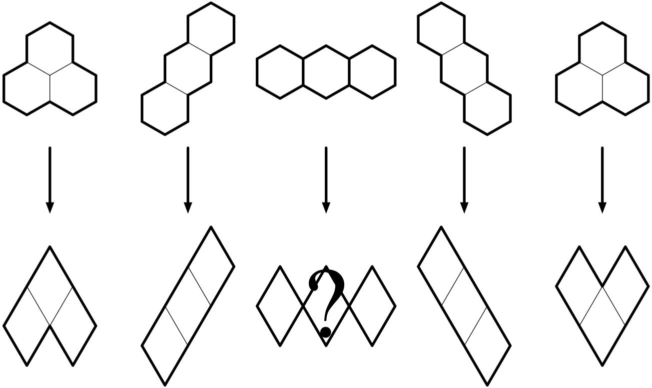
Connection to ribbon tilings

If we disallow one of the three orientations of bone tiles, then we're in the realm of ribbon tiles, and Sheffield's result applies.



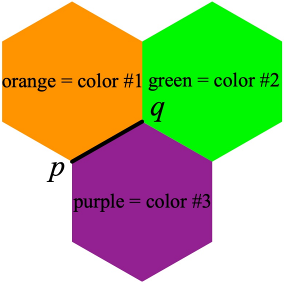
Connection to ribbon tilings

But if we allow all five (translationally-distinct) prototiles, the connection to ribbon tiles breaks down.



The correct definition of height functions?

A natural kind of height function for stones-and-bones tilings is given by a triple of integers that goes up by $e_i - e_j$ when you travel along a tile-edge with color i to your left and color j to your right (with $e_1 = (1, 0, 0)$ etc.).



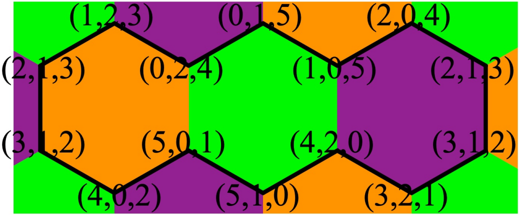
$$\theta(q) = \theta(p) + (1, 0, -1)$$

Stones-and-bones height functions

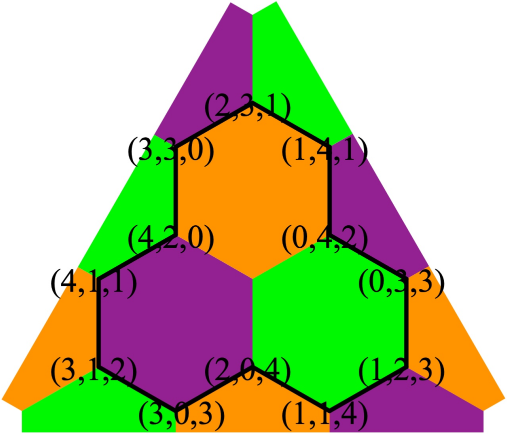
This height function is locally consistent because each tile is color-balanced, and globally consistent because of local consistency and simply-connectedness of the region.

It also retains all the information in the tiling: two vertices p, q that are adjacent in the underlying cell-complex of hexagons are joined by an edge in the tiling t if and only if the corresponding height function θ has the property that $||\theta(p) - \theta(q)||_1$ is 2, where $|| \cdot ||_1$ is the L_1-norm.

Consistency

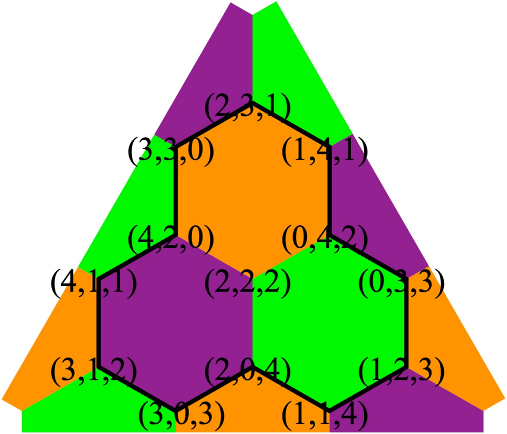


Consistency



Consistency

We assign each vertex in the interior of a stone a height equal to the average of the heights of its three neighbors.



Stones-and-bones height functions

All triples (a, b, c) that occur as heights of vertices have the same value of $a + b + c$, so our three-dimensional heights are really two-dimensional.

It's conceptually convenient to take $a + b + c = 0$ but it's typographically preferable to choose heights so that all integers that occur are non-negative (as in the figures).

It can be easily checked that if t_1 and t_2 are related by a Type I or Type II move, $\sum_p ||\theta_1(p) - \theta_2(p)||_1 = 36$.

So for general t_1 and t_2, by the triangle inequality, the (conceivably infinite!) moves-distance between the two tilings is bounded below by the sum

$$\frac{1}{36} \sum_p ||\theta_1(p) - \theta_2(p)||_1$$

A new, stronger conjecture

Conjecture (2024): For stones-and-bones tilings t_1, t_2 of a simply-connected region, the moves-distance $d(t_1, t_2)$ is given by the formula

$$d(t_1, t_2) = \frac{1}{36} \sum_p ||\theta_1(p) - \theta_2(p)||_1$$

That is, the height-function bound on moves-distance is tight.

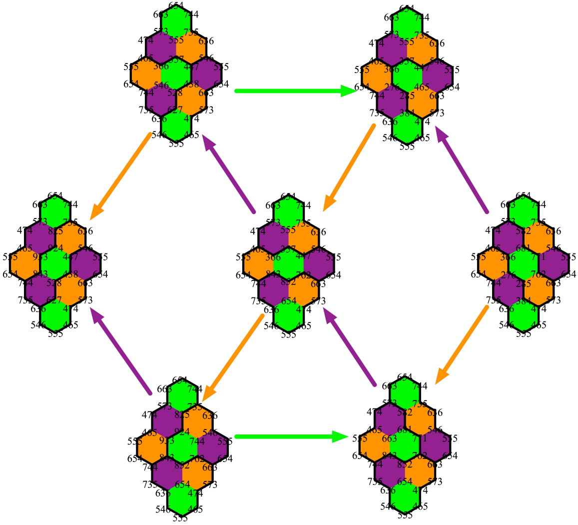
Ample experimental evidence supports the conjecture.

Six acyclic digraphs

Based on what each Type I and Type II move does to the height-triple, we assign each move a color as well as a direction. E.g., a "Green-ifying" move increases the Green component of the height-triple at the expense of the Orange and Purple components, while a "de-Green-ifying" move increases the Orange and Purple components at the expense of the Green component. Likewise for the other colors.

Six acyclic digraphs

Each edge of the moves-graph gets an orientation and a color.



Six acyclic digraphs

Define six sets of directed edges O^+, O^-, G^+, G^-, P^+, and P^- as follows:

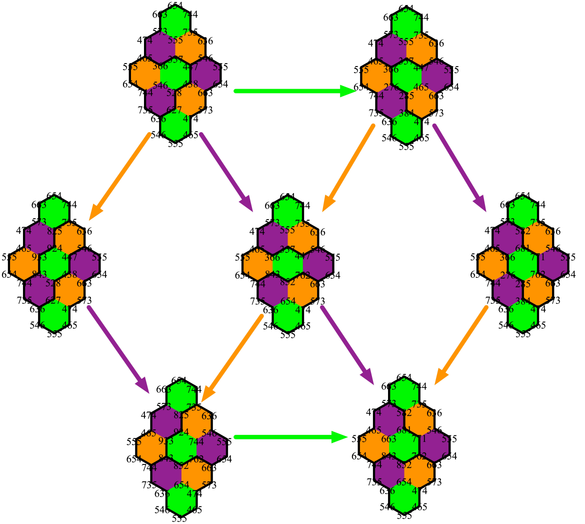
O^+ is the set of Orange-ifying directed edges, O^- is the set of de-Orange-ifying directed edges, etc.

Let $O^+G^+P^-$ be the directed graph associated with the union of O^+, G^+, and P^-, and likewise for the other choices of signs.

It's easy to see that $O^+G^+P^+$ and $O^-G^-P^-$ have cycles; it's not much harder to use height-functions to show that the other six graphs are acyclic.

Six acyclic digraphs

E.g., here's $O^+G^+P^-$:



Six acyclic digraphs

Conjecture (2025): For any simply-connected region, the six aforementioned acyclic digraphs are confluent. That is, there is only one sink-vertex, and every vertex has a path to that vertex.

Like the 2024 conjecture, the 2025 conjecture would imply the "old" conjecture (connectedness of the moves graph).

A simpler height function?

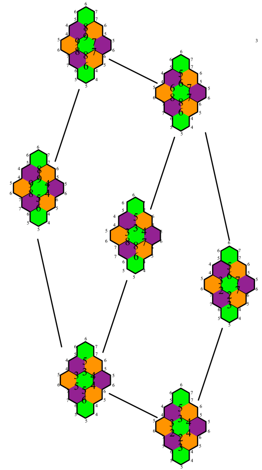
We've also looked at replacing the height-triple by just one of its three components.

Surprisingly, this stripped-down height function provably determines the tiling.

This height function is not however a height-function of the classical kind; in particular, these height-functions do not form a distributive lattice.

Here's the "Orange-ness" poset from the previous example:

A simpler height function?

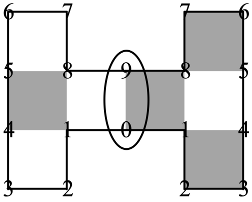


Might these posets always be lattices?

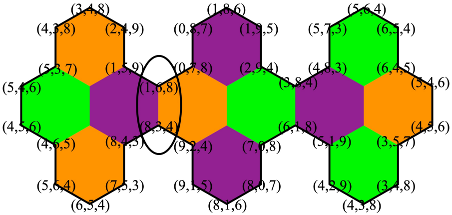
The tileability problem

Thurston used height functions to give a linear time algorithm for determining whether a region could or could not be tiled by dominoes or lozenges (here "linear" means "linear in area").

Can we do something similar for stones-and-bones tilings?



Difference = 9



Distance = 14

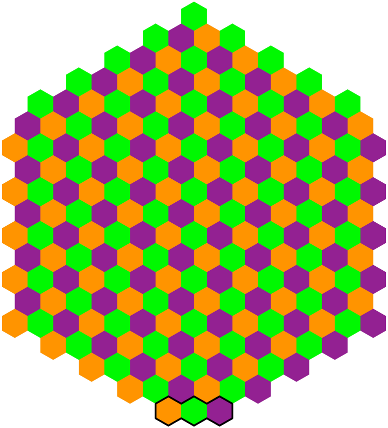
Random stones-and-bones tilings

David desJardins' highly efficient TilingCount program (described in his talk "Counting Tilings by Enlightened Brute Force"; here are links to his slides and video) can be used to compute $N(R)$, where R is any not-too-big simply-connected region in the hexagonal grid and $N(R)$ is the number of stones-and-bones tilings of R.

Consulted twice, it can compute the ratio $N(R \setminus t)/N(R)$ where t is a specific tile in R; this ratio is the probability that a uniformly random tiling of R contains the tile t.

Random stones-and-bones tilings

Here for instance is a specific region R and a specific tile at the bottom of R that's present in over 99% of the tilings of R.



There appear to be frozen regions in three of the corners.

Random stones-and-bones tilings

The right kind of height function might enable us to efficiently sample from the uniform distribution on stones-and-bones tilings of large (simply-connected) regions via some form of Coupling-From-The-Past, as we can do for domino tilings.

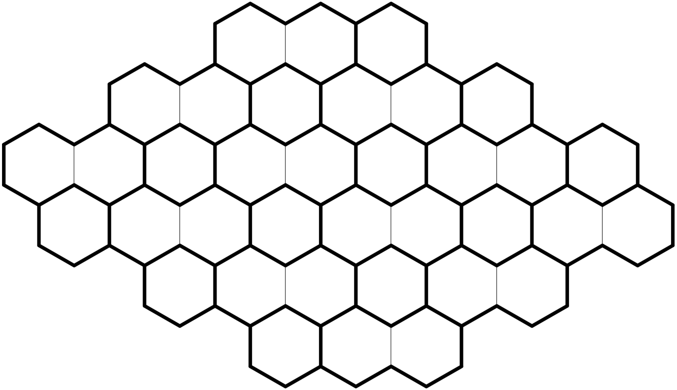
This might lead us to discover an analogue of the arctic circle theorem for the sorts of region I call benzels, introduced to serve as an analogue of Aztec diamonds in the stones-and-bones setting.

The page http://faculty.uml.edu/jpropp/benzels.html has many relevant links, including slides and videos from earlier talks I've given about stones-and-bones tilings.

Connection to dimer model on square grid

I just realized this week that the dimer model on a square grid "lives inside" this trimer model.

For instance, the stones and bones tilings of this region reduce to domino tilings of the Aztec diamond of order 3.



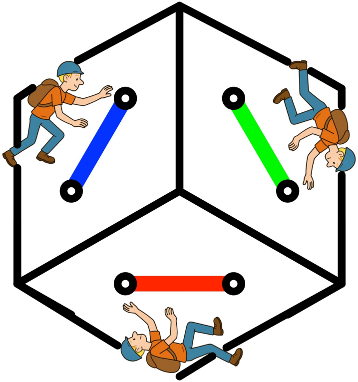
The second-to-last slide of this talk

All slides from my talk are available at

http://faculty.uml.edu/jpropp/kenyon.pdf

Thanks to the organizers for this wonderful conference!
And...

Happy birthday, Rick!



Thank you for your many beautiful perspectives on perfect matchings!