Combinatorial Ergodicity: Actions and Averages

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Slides for this talk are on-line at

http://jamespropp.org/mitcomb13a.pdf

Acknowledgments

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Mike LaCroix wrote fantastic postscript code to generate animations and pictures that illustrate our maps operating on order ideals on products of chains.

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Overview

For many actions τ on a finite set S of combinatorial objects, and for many natural real-valued statistics ϕ on S, one finds that the ergodic average

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} \phi(\tau^i(x))$$

is **independent** of the starting point $x \in S$.

We say that ϕ is **homomesic** (from Greek: "same middle") with respect to the combinatorial dynamical system (S, τ).

I'll give numerous examples of **homomesies** (homomesic functions), some proved and others conjectural.

Please interrupt with questions!

Introductory examples

1. Rotation of bit-strings
2. Bulgarian solitaire
3. Promotion of Near-Standard Young Tableaux (conjectural)
4. Suter's symmetries

Example 1: Rotation of bit-strings

Prop.: Let $\mathcal{O}$ be an orbit in the set of words w composed of a 0's and b 1's under the action of rotation (cyclic shift). Then

$$\frac{1}{\#\mathcal{O}} \sum_{w \in \mathcal{O}} \mathrm{inv}(w) = \frac{ab}{2}$$

where $\mathrm{inv}(w) = \#\{i, j : \; i < j, \; w_i > w_j\}$.

(E.g., $(\mathrm{inv}(0011) + \mathrm{inv}(0110) + \mathrm{inv}(1100) + \mathrm{inv}(1001))/4 = (0 + 2 + 4 + 2)/4 = 2 = \frac{(2)(2)}{2}$.)

I know two simple ways to prove this: one can show pictorially that the value of the sum doesn't change when you mutate w (replacing a 01 somewhere in w by 10 or vice versa), or one can write the number of inversions in w as $\sum_{i<j} w_i(1 - w_j)$ and then perform algebraic manipulations.

Example 2: Bulgarian solitaire

Given a way of dividing n identical chips into one or more heaps (represented as a partition λ of n), define $\tau(\lambda)$ as the partition of n that results from removing a chip from each heap and putting all the removed chips into a new heap.

E.g., for $n = 8$, two trajectories are

$$53 \to 42\underline{2} \to \mathbf{\underline{3}311} \to \mathbf{\underline{4}22} \to \dots$$

and

$$62 \to 5\underline{2}1 \to 4\underline{3}1 \to \mathbf{\underline{3}32} \to \mathbf{\underline{3}221} \to \mathbf{\underline{4}211} \to \mathbf{\underline{4}31} \to \dots$$

(the new heaps are underlined).

Let $\phi(\lambda)$ be the number of parts of λ.
In the forward orbit of $\lambda = (5, 3)$, the average value of ϕ is

$$(4 + 3)/2 = 7/2;$$

in the forward orbit of $\lambda = (6, 2)$, the average value of ϕ is

$$(3 + 4 + 4 + 3)/4 = 14/4 = 7/2.$$

Bulgarian solitaire: homomesies

Prop.: If $n = k(k-1)/2 + j$ with $0 \leq j < k$, then for every partition λ of n, the ergodic average of ϕ on the forward orbit of λ is $k - 1 + j/k$.

($n = 8$ corresponds to $k = 4$, $j = 2$.)

So the number-of-parts statistic on partitions of n is homomesic under the Bulgarian solitaire map.

The same is true for the size of the largest part, the size of the second largest part, etc.

Ignoring transience

Since S is finite, every forward orbit is eventually periodic, and the ergodic average of ϕ for the forward orbit that starts at x is just the average of ϕ over the periodic orbit that x eventually goes into.

So an equivalent way of stating our main definition in this case is, ϕ is homomesic with respect to (S, τ) iff the average of ϕ over each periodic τ-orbit $\mathcal{O}$ is the same for all $\mathcal{O}$.

In the rest of this talk, we'll restrict attention to maps τ that are invertible on S, so transience is not an issue.

Example 3: Promotion of Near-Standard Young Tableaux

Given a positive integer N, define a Near-Standard Young Tableau (NSYT) with "ceiling" N as a Young tableau T in which entries are distinct integers between 1 and N.

(When N equals the number of cells of T, this is just the definition of a Standard Young Tableau.)

For each $1 \leq i \leq N-1$, let s_i be the action on NSYT's with ceiling N that replaces i (if it occurs in T) by $i+1$, and vice versa, provided that this does not violate the weak-increase condition in the definition of Young tableaux, and let ∂ be the composition of the maps $s_1, s_2, \ldots, s_{N-1}$. This generalizes promotion of SYT's.

A small example of promotion

(taken from J. Striker and N. Williams, *Promotion and Rowmotion*, European J. Combin. 33 (2012), no. 8, 1919–1942; http://arxiv.org/abs/1108.1172):

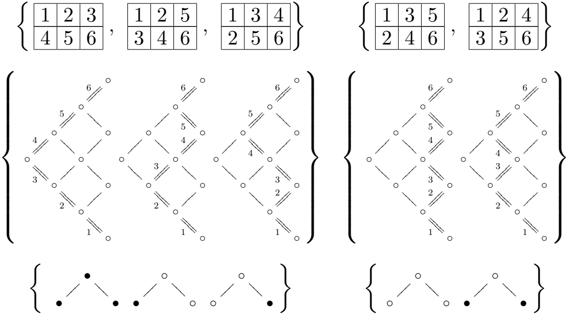


FIGURE 5. The two orbits of SYT of shape $(3,3)$ under promotion, the san
orbits using the maximal chain interpretation, and the same two orbits usi
order ideal interpretation.

A small example of promotion: centrally symmetric sums

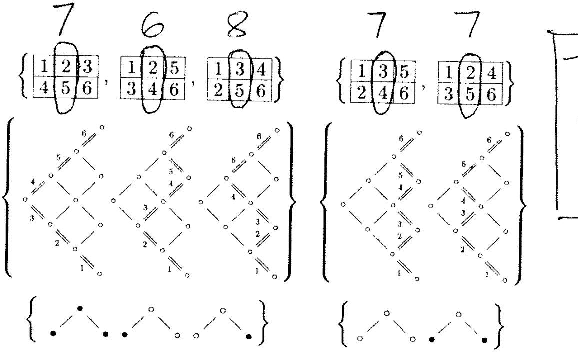


FIGURE 5. The two orbits of SYT of shape (3, 3) under promotion, the same
orbits using the maximal chain interpretation, and the same two orbits using
order ideal interpretation.

We apply this idea of boundary paths under ρ to noncrossing objects under rotation i
and generalize it in Section 7. In Sections 7 and 8, we conjecture that there is a further gen
to the type D_n positive root poset, plane partitions, the ASM poset, and the TSSCPP

5. THE CONJUGACY OF PROMOTION AND ROWMOTION

Promotion of Near-Standard Young Tableaux: homomesies

Conjecture: If c and c' are opposite cells of a Near-Standard Young Tableau T of rectangular shape λ,
i.e., c and c' are related by 180-degree rotation about the center (note: the case $c = c'$ is permitted when λ is odd-by-odd),
and $\phi(T)$ denotes the sum of the numbers in cells c and c',
then ϕ is homomesic under ∂ with average value $N + 1$.

Addendum (added a few weeks after the talk)

The Bender-Knuth involutions are operations on column-strict tableaux that generalize the maps $s_1, \ldots, s_{N-1}$ discussed above: If a tableau has a i's and b $i+1$'s, then after the ith Bender-Knuth involution is applied, the resulting tableau has b i's and a $i+1$'s. One can define promotion on column-strict skew tableaux with ceiling N by successively applying the ith Bender-Knuth involution, with i going from 1 to $N-1$.

Conjecture: If the shape of a skew tableau has central symmetry, and $\phi(T)$ denotes the sum of the numbers in cells c and c' where cells c and c' are opposite one another, then ϕ is homomesic under promotion with average value $N+1$.

This is known when λ has one row or one column.

Example 4: Suter's symmetries

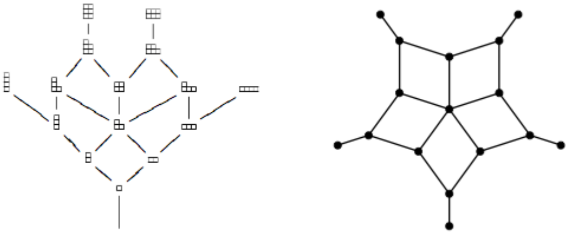
Let $\mathbb{Y}_N$ be the set of number-partitions λ whose maximal hook lengths are strictly less than N (i.e., whose Young diagrams fit inside some rectangle that fits inside the staircase shape $(N-1, N-2, ..., 2, 1)$).

Suter showed that the Hasse diagram of $\mathbb{Y}_N$ has N-fold cyclic symmetry (indeed, N-fold dihedral symmetry) by exhibiting an explicit action of order N.

Suter's action, $N = 5$

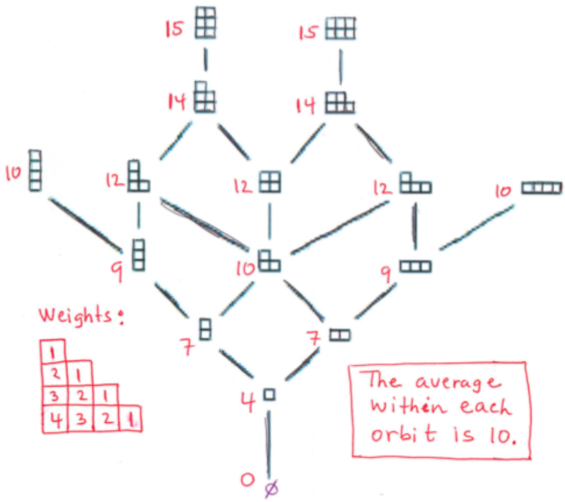
(taken from R. Suter, *Young's lattice and dihedral symmetries revisited: Möbius strips and metric geometry*; http://arxiv.org/abs/1212.4463):

Example (Hasse diagram of $\mathbb{Y}_5$ and its undirected Hasse graph)



This graph has a 5-fold (cyclic) symmetry, and its full symmetry group is a dihedral group of order 10.

Suter's action, $N = 5$: weighted sums



Suter's action: homomesies

Assign weight 1 to the cells at the diagonal boundary of the staircase shape, weight 2 to their neighbors, ..., and weight $N-1$ to the cell at the lower left, and for $\lambda \in \mathbb{Y}_N$ let $\phi(\lambda)$ be the sum of the weights of all the cells in the Young diagram of λ.

Prop. (Einstein, P.): ϕ is homomesic under Suter's map with average value $(n^3 - n)/12$.

More refined result: If $i + j = N$ (note: $i = j$ is permitted), and $\phi_{i,j}(\lambda)$ is the sum of the weights of all the cells in λ with weight i plus the sum of the weights of all the cells in λ with weight j, then $\phi_{i,j}$ is homomesic under Suter's map with average ij in all orbits.

The main part of the talk

The Panyushev complement

Antichains in $[a] \times [b]$

Order ideals in $[a] \times [b]$

An invertible operation on antichains

Let $\mathcal{A}(P)$ be the set of antichains of a finite poset P.

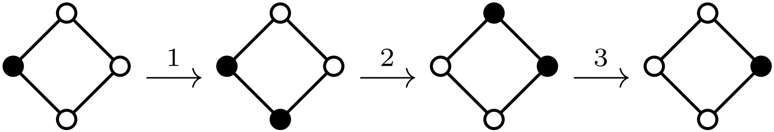
Given $A \in \mathcal{A}(P)$, let $\tau(A)$ be the set of minimal elements of the complement of the downward-saturation of A.
τ is invertible since it is a composition of three invertible operations:

$$\text{antichains} \longleftrightarrow \text{downsets} \longleftrightarrow \text{upsets} \longleftrightarrow \text{antichains}$$

This map and its inverse have been considered with varying degrees of generality, by many people more or less independently (using a variety of nomenclatures and notations): Duchet, Brouwer and Schrijver, Cameron and Fon Der Flaass, Fukuda, Panyushev, Rush and Shi, and Striker and Williams.

An example

1. Saturate downward
2. Complement
3. Take minimal element(s)



(For a bigger example, see the example of rowmotion on slide 4 of `http://www.math.uconn.edu/~troby/combErg2012kizugawa.pdf`.)

Panyushev's conjecture

Let Δ be a reduced irreducible root system in $\mathbf{R}^n$.
Choose a system of positive roots and make it a poset of rank n by decreeing that y covers x iff $y - x$ is a simple root.
Conjecture (Conjecture 2.1(iii) in D.I. Panyushev, *On orbits of antichains of positive roots*, European J. Combin. 30 (2009), 586-594): Let $\mathcal{O}$ be an arbitrary τ-orbit. Then

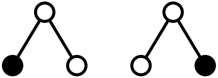
$$\frac{1}{\#\mathcal{O}} \sum_{A \in \mathcal{O}} \#(A) = \frac{n}{2}.$$

(Two other assertions of this kind, Panyushev's Conjectures 2.3(iii) and 2.4(ii), appear to remain open.)

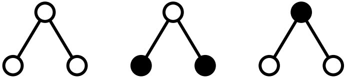
Panyushev's Conjecture 2.1(iii) (along with much else) was proved by Armstrong, Stump, and Thomas in their article *A uniform bijection between nonnesting and noncrossing partitions*, http://arxiv.org/abs/1101.1277.

Panyushev's conjecture: The A_n case, $n = 2$

Here we have just an orbit of size 2 and an orbit of size 3:



$1 \qquad 1$



$0 \qquad 2 \qquad 1$

Within each orbit, the average antichain has cardinality $n/2 = 1$.

Antichains in $[a] \times [b]$: cardinality is homomesic

A simpler-to-prove phenomenon of this kind concerns the poset $[a] \times [b]$ (where $[k]$ denotes the linear ordering of $\{1, 2, \ldots, k\}$):

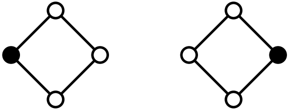
Theorem (P., Roby): Let $\mathcal{O}$ be an arbitrary τ-orbit in $\mathcal{A}([a] \times [b])$. Then

$$\frac{1}{\#\mathcal{O}} \sum_{A \in \mathcal{O}} \#(A) = \frac{ab}{a+b}.$$

This is an easy consequence of unpublished work of Hugh Thomas building on earlier work of Richard Stanley: see the last paragraph of section 2 of R. Stanley, *Promotion and evacuation*, http://www.combinatorics.org/ojs/index.php/eljc/article/view/v16i2r9 .

Antichains in $[a] \times [b]$: the case $a = b = 2$

Here we have an orbit of size 2 and an orbit of size 4:



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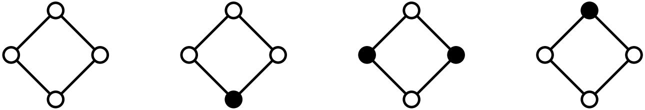
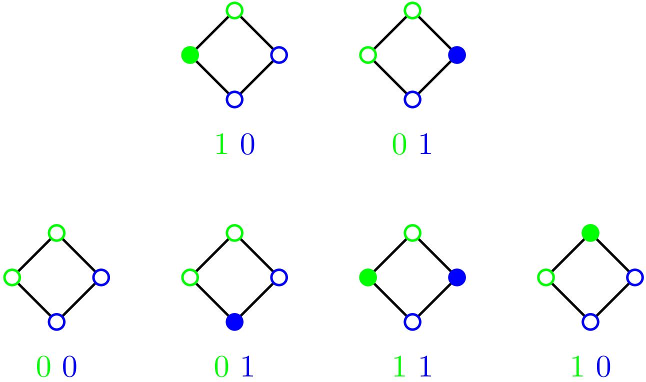


$0 \qquad 1 \qquad 2 \qquad 1$

Within each orbit, the average antichain has cardinality $ab/(a+b) = 1$.

Antichains in $[a] \times [b]$: fiber-cardinality is homomesic



Within each orbit, the average antichain has
$1/2$ a green element and $1/2$ a blue element.

Antichains in $[a] \times [b]$: fiber-cardinality is homomesic

For $(i,j) \in [a] \times [b]$, and A an antichain in $[a] \times [b]$, let $1_{i,j}(A)$ be 1 or 0 according to whether or not A contains (i,j).

Also, let $f_i(A) = \sum_{j \in [b]} 1_{i,j}(A) \in \{0,1\}$ (the cardinality of the intersection of A with the fiber $\{(i,1),(i,2),\ldots,(i,b)\}$ in $[a] \times [b]$), so that $\#(A) = \sum_i f_i(A)$.
Likewise let $g_j(A) = \sum_{i \in [a]} 1_{i,j}(A)$, so that $\#(A) = \sum_j g_j(A)$.

Theorem (P., Roby): For all i, j,

$$\frac{1}{\#\mathcal{O}} \sum_{A \in \mathcal{O}} f_i(A) = \frac{b}{a+b} \qquad \text{and} \qquad \frac{1}{\#\mathcal{O}} \sum_{A \in \mathcal{O}} g_j(A) = \frac{a}{a+b}.$$

The indicator functions f_i and g_j are homomesic under τ, even though the indicator functions $1_{i,j}$ aren't.

Antichains in $[a] \times [b]$: centrally symmetric homomesies

Theorem (P., Roby): In any orbit, the number of A that contain (i,j) equals the number of A that contain the opposite element $(i',j') = (a+1-i, b+1-j)$.

That is, the function $1_{i,j} - 1_{i',j'}$ is homomesic under τ, with average value 0 in each orbit.

Linearity

Useful triviality: every linear combination of homomesies is itself homomesic.

E.g., consider the adjusted major index statistic defined by $\mathrm{amaj}(A) = \sum_{(i,j)\in A}(i-j)$.

P. and Roby proved that amaj is homomesic under τ
by writing it as a linear combination of the functions $1_{i,j} - 1_{i',j'}$.
Haddadan gave a simpler proof,
writing amaj as a linear combination of the functions f_i and g_j.

Question: Are there other homomesic combinations of the indicator functions $1_{i,j}$ (with $(i,j) \in [a] \times [b]$),
linearly independent of the functions f_i, g_j, and $1_{i,j} - 1_{i',j'}$?

From antichains to order ideals

Given a poset P and an antichain A in P, let $\mathcal{I}(A)$ be the order ideal $I = \{y \in P\ : y \leq x \text{ for some } x \in A\}$ associated with A, so that for any order ideal I in P, $\mathcal{I}^{-1}(I)$ is the antichain of maximal elements of I.

As usual, we let $J(P)$ denote the set of (order) ideals of P.

We define $\overline{\tau} : J(P) \to J(P)$ by $\overline{\tau}(I) = \mathcal{I}(\tau(\mathcal{I}^{-1}(I)))$. That is, $\overline{\tau}(I)$ is the downward saturation of the set of minimal elements of the complement of I.

For $(i,j) \in P$ and $I \in J(P)$, let $\overline{1}_{i,j}(I)$ be 1 or 0 according to whether or not I contains (i,j).

One action, two vector spaces

$\overline{\tau}$ is "the same" τ in the sense that the standard bijection from $\mathcal{A}(P)$ to $J(P)$ (downward saturation) makes the following diagram commute:

$$\begin{array}{ccc} \mathcal{A}(P) & \xrightarrow{\tau} & \mathcal{A}(P) \\ \downarrow & & \downarrow \\ J(P) & \xrightarrow{\overline{\tau}} & J(P) \end{array}$$

However, the bijection from $\mathcal{A}(P)$ to $J(P)$ does **not** carry the vector space generated by the functions $1_{i,j}$ to the vector space generated by the functions $\overline{1}_{i,j}$ in a linear way.

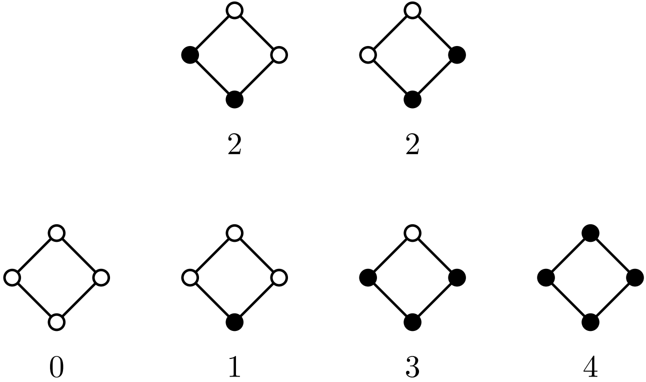
So the homomesy situation for $\overline{\tau} : J(P) \rightarrow J(P)$ could be (and, as we'll see, is) different from the homomesy situation for $\tau : \mathcal{A}(\mathcal{P}) \rightarrow \mathcal{A}(\mathcal{P})$.

Ideals in $[a] \times [b]$: cardinality is homomesic

Theorem (P., Roby): Let $\mathcal{O}$ be an arbitrary $\overline{\tau}$-orbit in $J([a] \times [b])$. Then
$$\frac{1}{\#\mathcal{O}} \sum_{I \in \mathcal{O}} \#(I) = \frac{ab}{2}.$$

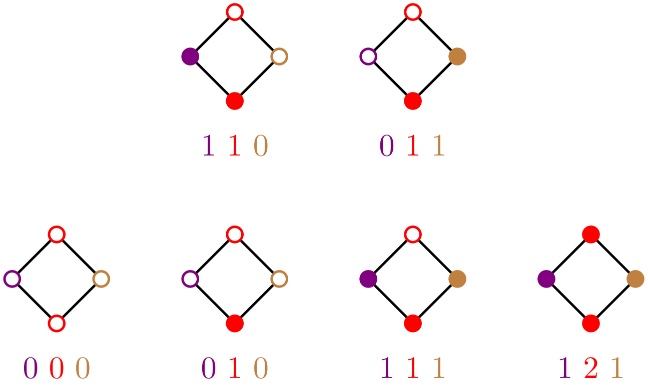
Ideals in $[a] \times [b]$: the case $a = b = 2$

Again we have an orbit of size 2 and an orbit of size 4:



Within each orbit, the average order ideal has cardinality $ab/2 = 2$.

Ideals in $[a] \times [b]$: file-cardinality is homomesic



Within each orbit, the average order ideal has
$1/2$ a violet element, 1 red element, and $1/2$ a brown element.

Ideals in $[a] \times [b]$: file-cardinality is homomesic

For $1-b \leq k \leq a-1$, define the kth **file** of $[a] \times [b]$ as

$$\{(i,j) : 1 \leq i \leq a,\ 1 \leq j \leq b,\ i-j=k\}.$$

For $1-b \leq k \leq a-1$, let $h_k(I)$ be the number of elements of I in the kth file of $[a] \times [b]$, so that $\#(I) = \sum_k h_k(I)$.

Theorem (P., Roby): For every $\overline{\tau}$-orbit $\mathcal{O}$ in $J([a] \times [b])$,

$$\frac{1}{\#\mathcal{O}} \sum_{I \in \mathcal{O}} h_k(I) = \begin{cases} \frac{(a-k)b}{a+b} & \text{if } k \geq 0 \\ \frac{a(b+k)}{a+b} & \text{if } k \leq 0. \end{cases}$$

Ideals in $[a] \times [b]$: centrally symmetric homomesies

Recall that for $(i,j) \in [a] \times [b]$, and I an ideal in $[a] \times [b]$, $\overline{1}_{i,j}(I)$ is 1 or 0 according to whether or not I contains (i,j).

Write $(i',j') = (a+1-i, b+1-j)$, the point opposite (i,j) in the poset.

Theorem (P., Roby): $\overline{1}_{i,j} + \overline{1}_{i',j'}$ is homomesic under $\overline{\tau}$.

Question: In addition to the functions h_k and $\overline{1}_{i,j} + \overline{1}_{i',j'}$, are there other homomesic functions in the span of the functions $\overline{1}_{i,j}$?

The two vector spaces, compared

In the space associated with antichains:

fiber-cardinalities and

centrally symmetric **differences**

are homomesic.

In the space associated with order ideals:

file-cardinalities and

centrally symmetric **sums**

are homomesic.

Extra topics

Toggling

Other actions

Other posets

Continuous piecewise-linear maps

Non-periodic actions

Toggling

In their 1995 article *Orbits of antichains revisited*, European J. Combin. 16 (1995), 545–554, Cameron and Fon-der-Flaass give an alternative description of $\overline{\tau}$.

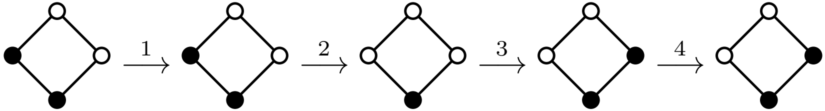
Given $I \in J(P)$ and $x \in P$, let $\tau_x(I) = I \triangle \{x\}$ provided that $I \triangle \{x\}$ is an order ideal of P; otherwise, let $\tau_x(I) = I$.

We call the involution τ_x "toggling at x".

The involutions τ_x and τ_y commute unless x covers y or y covers x.

An example

1. Toggle the top element
2. Toggle the left element
3. Toggle the right element
4. Toggle the bottom element



Toggling from top to bottom

Theorem (Cameron and Fon-der-Flaass): Let $x_1, x_2, \ldots, x_n$ be any order-preserving enumeration of the elements of the poset P. Then the action on $J(P)$ given by the composition $\tau_{x_1} \circ \tau_{x_2} \circ \cdots \circ \tau_{x_n}$ coincides with the action of $\overline{\tau}$.

In the particular case $P = [a] \times [b]$, we can enumerate P rank-by-rank; that is, we can list the (i, j)'s in order of increasing $i + j$.

Note that all the involutions coming from a given rank of P commute with one another, since no two of them are in a covering relation.

Striker and Williams refer to $\overline{\tau}$ (and τ) as **rowmotion**, since for them, "row" means "rank".

Toggling from side to side

Recall that a file in $P = [a] \times [b]$ is the set of all $(i, j) \in P$ with $i - j$ equal to some fixed value k.

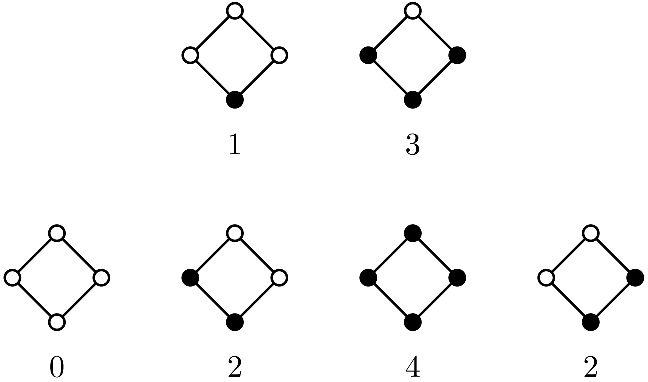
Note that all the involutions coming from a given file commute with one another, since no two of them are in a covering relation.

It follows that for any enumeration $x_1, x_2, \ldots, x_n$ of the elements of the poset $[a] \times [b]$ arranged in order of increasing $i - j$, the action on $J(P)$ given by $\tau_{x_1} \circ \tau_{x_2} \circ \cdots \circ \tau_{x_n}$ doesn't depend on which enumeration was used.

Striker and Williams call this well-defined composition **promotion**, and denote it by ∂, since it is closely related to Schützenberger's notion of promotion on linear extensions of posets.

Promoting ideals in $[a] \times [b]$: the case $a = b = 2$

Again we have an orbit of size 2 and an orbit of size 4:



$J([a] \times [b])$: cardinality is homomesic under promotion

Claim (P., Roby): Let $\mathcal{O}$ be an arbitrary orbit in $J([a] \times [b])$ under the action of promotion ∂. Then

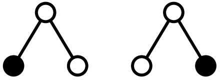
$$\frac{1}{\#\mathcal{O}} \sum_{I \in \mathcal{O}} \#(I) = \frac{ab}{2}.$$

The result about cyclic rotation of binary words discussed earlier ("Example 1") turns out to be a special case of this.

Root posets of type A: antichains

Recall that, by the Armstrong-Stump-Thomas theorem, the cardinality of antichains is homomesic under the action of rowmotion, where the poset P is a root poset of type A_n.

E.g., for $n = 2$:



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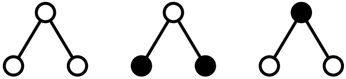


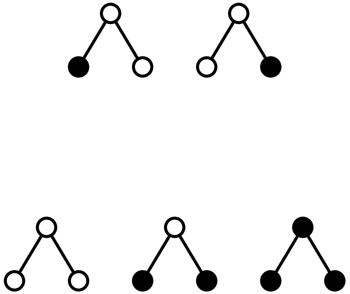
$0 \qquad 2 \qquad 1$

Antichain-cardinality is homomesic: in each orbit, its average is 1.

Root posets of type A: order ideals

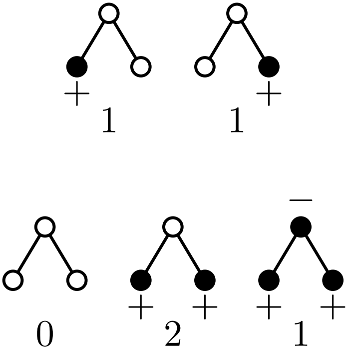
What if instead of antichains we take order ideals?

E.g., $n = 2$:



What is homomesic here?

Root posets of type A: rank-signed cardinality



Root posets of type A: rank-signed cardinality is homomesic

Theorem (Haddadan): Let P be the root poset of type A_n. If we assign an element $x \in P$ weight $\mathrm{wt}(x) = (-1)^{\mathrm{rank}(x)}$, and assign a order ideal $I \in J(P)$ weight $\phi(I) = \sum_{x \in I} \mathrm{wt}(x)$, then ϕ is homomesic under rowmotion and promotion, with average $n/2$.

The order polytope of a poset

Let P be a poset, with an extra minimal element $\hat{0}$ and an extra maximal element $\hat{1}$ adjoined.

The order polytope $\mathcal{O}(P)$ (introduced by R. Stanley) is the set of functions $f : P \to [0,1]$ with $f(\hat{0}) = 0$, $f(\hat{1}) = 1$, and $f(x) \le f(y)$ whenever $x \le_P y$.

Flipping-maps in the order polytope

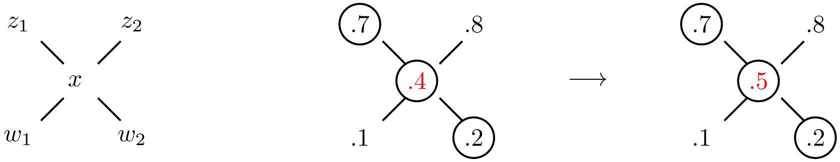
For each $x \in P$, define the flip-map $\sigma_x : \mathcal{O}(P) \to \mathcal{O}(P)$ sending f to the unique f' satisfying

$$f'(y) = \begin{cases} f(y) & \text{if } y \neq x, \\ \min_{z \cdot > x} f(z) + \max_{w < \cdot x} f(w) - f(x) & \text{if } y = x, \end{cases}$$

where $z \cdot > x$ means z covers x and $w < \cdot x$ means x covers w.

Note that the interval $[\min_{z \cdot > x} f(z), \max_{w < \cdot x} f(w)]$ is precisely the set of values that $f'(x)$ could have so as to satisfy the order-preserving condition, if $f'(y) = f(y)$ for all $y \neq x$; the map that sends $f(x)$ to $\min_{z \cdot > x} f(z) + \max_{w < \cdot x} f(w) - f(x)$ is just the affine involution that swaps the endpoints.

Example



$$\min_{z \cdot> x} f(z) + \max_{w <\cdot x} f(w) = .7 + .2 = .9$$

$$f(x) + f'(x) = .4 + .5 = .9$$

Flipping and toggling

If we associate each order-ideal I with the indicator function of $P \setminus I$ (that is, the function that takes the value 0 on I and the value 1 everywhere else), then toggling I at x is tantamount to flipping f at x.

That is, we can identify $J(P)$ with the vertices of the polytope $\mathcal{O}(P)$ in such a way that toggling can be seen to be a special case of flipping.

This may be clearer if you think of $J(P)$ as being in bijection with the set of monotone 0,1-valued functions on P.

Flipping

Flipping (at least in special cases) is not new, though it is not well-studied; the most worked-out example I've seen is Berenstein and Kirillov's article *Groups generated by involutions, Gelfand-Tsetlin patterns and combinatorics of Young tableaux* (St. Petersburg Math. J. 7 (1996), 77–127); see http://pages.uoregon.edu/arkadiy/bk1.pdf.

Composing flips

Just as we can apply toggle-maps from top to bottom, we can apply flip-maps from top to bottom:

$$
\begin{array}{ccc} & .8 & \\ .4 & & .3 \\ & .1 & \end{array}
\qquad
\begin{array}{ccc} & .6 & \\ .4 & & .3 \\ & .1 & \end{array}
\qquad
\begin{array}{ccc} & .6 & \\ .3 & & .3 \\ & .1 & \end{array}
$$

$$
\begin{array}{ccc} & .6 & \\ .3 & & .4 \\ & .1 & \end{array}
\qquad
\begin{array}{ccc} & .6 & \\ .3 & & .4 \\ & .2 & \end{array}
$$

(Here we successively flip values at the North, West, East, and South.)

Conjectures

It appears that all of the aforementioned results on homomesy for rowmotion and promotion on $J([a] \times [b])$ lift to corresponding results in the order polytope, where instead of composing toggle-maps to obtain rowmotion and promotion we compose the corresponding flip-maps to obtain continuous piecewise-linear maps from $\mathcal{O}([a] \times [b])$ to itself.

The first step would be to show that rowmotion and promotion on $\mathcal{O}([a] \times [b])$, defined as above, are maps of order $a + b$.

Example

An orbit of (lifted) rowmotion (flipping values from top to bottom):

$$
\begin{array}{ccc}
 & .7 & \\
.2 & & .4 \\
 & .1 &
\end{array}
\qquad
\begin{array}{ccc}
 & .7 & \\
.6 & & .4 \\
 & .3 &
\end{array}
\qquad
\begin{array}{ccc}
 & .9 & \\
.6 & & .8 \\
 & .3 &
\end{array}
\qquad
\begin{array}{ccc}
 & .9 & \\
.6 & & .4 \\
 & .1 &
\end{array}
$$

The average is

$$
\begin{array}{ccc}
 & .8 & \\
.5 & & .5 \\
 & .2 &
\end{array}
$$

Continuous piecewise-linear maps

Not only does the polytope perspective allow us to see toggling as the restriction of a continuous piecewise-linear (c.p.l.) map, but it also lets us see the bijection from $J(P)$ to $\mathcal{A}(P)$ as the restriction of the c.p.l. map $f \mapsto g$ where $g(x) = \min_{y \cdot > x}(f(y) - f(x))$ (this is Stanley's bijection between the order polytope and the chain polytope).

This allows us to lift rowmotion on $\mathcal{A}(P)$ to a polytope action, and preliminary experiments suggest that all of the results on homomesy for rowmotion on $\mathcal{A}([a] \times [b])$ lift to corresponding results in a polytope.

Example (continued):

$$
\begin{array}{ccc} & .3 & \\ .5 & & .3 \\ & .1 & \end{array}
\qquad
\begin{array}{ccc} & .3 & \\ .1 & & .3 \\ & .1 & \end{array}
\qquad
\begin{array}{ccc} & .1 & \\ .3 & & .1 \\ & .3 & \end{array}
\qquad
\begin{array}{ccc} & .1 & \\ .3 & & .5 \\ & .3 & \end{array}
$$

The average is

$$
\begin{array}{ccc} & .2 & \\ .3 & & .3 \\ & .2 & \end{array}
$$

The c.p.l. category seems likely to prove to be the "right" setting for many of these results.

Invariants

In this talk I've stressed the **homomesies** of dynamical systems (S, τ), but equally important are the **invariants** of such systems: real-valued functions f on S s.t. $f(\tau(x)) = f(x)$ for all $x \in S$.

Like homomesies, invariants in dynamical algebraic combinatorics are often naturally viewed in the piecewise-linear setting, as in the Berenstein-Kirillov paper.

E.g., under the action of rowmotion on $\mathcal{O}(P)$ with $P = [a] \times [b]$,

$$\min_{x,y \in P:\ y \cdot> x} (f(y) - f(x))$$

appears to be invariant.

Homomesies and invariants

When τ is not just piecewise-linear but actually linear, the space of homomesies and the space of invariants are **complementary**. See section 2.2 of http://jamespropp.org/propp-roby.pdf.

In the general case, where we are given a vector space of functions on S whose basis-elements correspond to combinatorial features of S, there is no guarantee that the homomesic subspace and invariant subspace will be complementary, but they are always nearly disjoint (the only functions that are both invariants and homomesies are constants).

The second-to-last conjecture of this talk

Let $P = [2] \times [2]$. One can show by brute force that the c.p.l. maps

$$\sigma_{(1,1)} \circ \sigma_{(1,2)} \circ \sigma_{(2,1)} \circ \sigma_{(2,2)}$$

("lifted rowmotion") and

$$\sigma_{(2,1)} \circ \sigma_{(1,1)} \circ \sigma_{(2,2)} \circ \sigma_{(1,2)}$$

("lifted promotion") are each of order 4.

Conjecture: The c.p.l. map

$$\sigma_{(1,1)} \circ \sigma_{(1,2)} \circ \sigma_{(2,2)} \circ \sigma_{(2,1)}$$

(flipping values in clockwise order, as opposed to going by rows or columns of P) is of infinite order.

The last conjecture of this talk

Conjecture: The homomesy results for $J([2] \times [2])$ apply here too. (Note: now the relevant notion of average is indeed an ergodic average, since the space no longer consists of finite orbits).

Note that taking $P = [2] \times [2]$ is just a way of getting our toe in the door; I expect $[a] \times [b]$ to exhibit similar behavior.

The last slide of this talk

I've found lots of examples of conjectural homomesies in all branches of combinatorics, starting at the level of the twelve-fold way and progressing through spanning trees, parking functions, abelian sandpiles (aka chip-firing), rotor-routing, etc.

I'd be glad to advise grad students and undergrads who want to work in this area.

For more information, see:

http://jamespropp.org/ucbcomb12.pdf
http://jamespropp.org/mathfest12a.pdf
http://www.math.uconn.edu/∼troby/combErg2012kizugawa.pdf
http://jamespropp.org/mitcomb13a.pdf
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