Sets with a Negative Number of Elements

D. Loeb

November 16, 1995

1 Introduction

Given a universe of discourse U a *multiset* can be thought of as a function M from U to the natural numbers $\mathbf{N}$. In this paper, we define a *hybrid set* to be any function from the universe U to the integers $\mathbf{Z}$. These sets are called hybrid since they contain elements with either a positive or negative multiplicity.

Our goal is to use these hybrid sets *as if* they were multisets in order to adequately generalize combinatorial facts which are true classically for only nonnegative integers. However, the definition above does not tell us much about these hybrid sets; we must define operations on them which provide us with the needed combinatorial structure.

We will define an analog of a set which can contain either a positive or negative number of elements. We will allow sums to be calculated over an arbitrary hybrid set. This will lead us to a generalization of symmetric functions to a negative number of variable which agrees with all previous known generalizations of symmetric functions in this direction. From these new symmetric functions, we calculate generalized binomial coefficients. Then these coefficients will be given a combinatorial interpretation in terms of a new partial order on the hybrid sets. Next, we will generalize linear partitions using this partial order and enumerate them. Finally, we define sum from one to an arbitrary integer and derive formulas involving them including a generalization of the Stirling numbers.

2 Hybrid Sets

One usually thinks of a multiset M as a function from some universe U to the natural numbers $\mathbf{N}$. $M(x)$ is called the *multiplicity* of x in the multiset M. Sets then are a special case of multisets in which all x have multiplicity either 1 or 0.

In this paper, we generalize these concepts to new "sets" and new "multisets" which may have negative integers as multiplicities as well as nonnegative integers.

Definition 2.1 (Hybrid Sets) *Given a universe U, any function $f : U \to \mathbf{Z}$ is called a* hybrid set*. The value of $f(x)$ is said to be the* multiplicity *of the element x. If $f(x) \neq 0$ we say x is a member of f and write $x \in f$; otherwise, we write $x \notin f$. Define the* number of elements $\#f$ *to be the sum $\sum_{x \in U} f(x)$. f is said to be an $\#f$ (element) hybrid set.*

We will denote hybrid sets by using the usual set braces $\{\}$ and inserting a bar in the middle $\{|\}$. Elements occuring with positive multiplicity are written on the left of the bar, and elements occuring with negative multiplicity are written on the right. Order is completely irrelevant. We will never have occasion to write the same element on both sides of the bar.

For example, if $f = \{a, b, c, b|d, e, e\}$ then $f(a) = 1$, $f(b) = 2$, $f(c) = 1$, $f(d) = -1$, $f(e) = -2$, and $f(x) = 0$ for $x \neq a, b, c, d, e$.

Definition 2.2 (New Sets) *A* positive *or* classical set *is a hybrid set taking only values 0 and 1. A* negative set *is a hybrid set taking only values 0 and −1. A* new set *is either a positive or a negative set.*

For example, the set $S = \{a, b, c\}$ can be written as the classical or positive set $S = \{b, a, c|\}$ while $-S = \{|a, b, c\}$ is a negative set. Both S and $-S$ are new sets. However, $\{a|b\}$ is not a new set.

The *empty set* $\emptyset = \{|\}$ is the unique hybrid set for which all elements have multiplicity zero. It is thus simultaneously a positive set and a negative set.

3 Sums and Products over a Hybrid Set

One often wishes to sum an expression over all the elements of a set. For example, we usually write

$$\phi(n) = \sum_{i \in F(n)} i$$

for the Euler phi function where $F(n)$ is the set of positive factors of an integer n.

When dealing with multisets, it is often convenient to include multiplicities in the summation. One defines $\sum_{x \in M} F(x)$ to be $\sum_{x \in U} M(x)F(x)$. Similarly, we can expand the characteristic polynomial of a matrix as

$$\prod_{\lambda \in E} (x - \lambda)$$

where E is its multiset of eigenvalues.

In general, for any function G and hybrid set f we write $\sum_{x \in f} G(x)$ for $\sum_{x \in U} f(x)G(x)$, and we write $\prod_{x \in f} G(x)$ for $\prod_{x \in U} G(x)^{f(x)}$.

As an application, we demonstrate that the elementary symmetric function in a “negative” number of variables is in fact the complete symmetric function.

Definition 3.1 (The Complete and Elementary Symmetric Functions) *1. The* complete symmetric function *of degree n over the set of variable V $h_n(V)$ is defined explicitly by the sum*

$$h_n(V) = \sum_M \prod_{x \in M} x$$

where the sum is over all n element multisets based on the set of variables V, and implicitly by the generating function

$$\prod_{x \in V} (1 - xy)^{-1} = \sum_{n \geq 0} h_n(V) y^n.$$

2. The elementary symmetric function *of degree n over the set V of variables $e_n(V)$ is defined explicitly by the sum*

$$e_n(V) = \sum_{\substack{S \subseteq V \\ \#S = n}} \prod_{x \in S} x,$$

and implicitly by the generating function

$$\prod_{x \in V} (1 + xy) = \sum_{n \geq 0} e_n(V) y^n.$$

Now, if V is the negative set of variables $\{|y_1, y_2, \ldots\}$, then

$$\begin{aligned} \sum_{n \geq 0} h_n(V) t^n &= \prod_{x \in V} (1 - xt)^{-1} \\ &= \prod_{k \geq 1} \left((1 - y_k t)^{-1}\right)^{-1} \\ &= \sum_{n \geq 0} e_n(-y_1, -y_2, \ldots) t^n, \end{aligned}$$

and

$$\begin{aligned} \sum_{n \geq 0} e_n(V) t^n &= \prod_{x \in V} (1 + xt) \\ &= \prod_{k \geq 1} (1 + y_k t)^{-1} \\ &= \sum_{n \geq 0} h_n(-y_1, -y_2, \ldots) t^n. \end{aligned}$$

Thus,

$$e_n(a, b, c, \ldots | x, y, z, \ldots) = h_n(-x, -y, -z, \ldots | -a, -b, -c, \ldots),$$

so it is immediate that the transformation

$$* : e_n \mapsto h_n$$

used in the theory of symmetric functions is an involution.

4 Roman Coefficients

When V is a classical set of n variables, $e_k(V)$ contains $\binom{n}{k}$ terms, so if we set all the variables to one, then we have the binomial coefficient:

$$e_k(V)_{\substack{x=1\\ \forall x\in V}} = \binom{n}{k}.$$

However, the left hand side of this equation is well defined even when n is a negative integer and V is a negative set. In this case, it can be used as a generalization of the binomial coefficients. This generalization is already well known, and it is usually

$$\binom{n}{k} = \frac{n(n-1)(n-2)\cdots(n-k+1)}{k!}.$$

However, this is only well defined for k nonnegative; using the following result we will permit both n and k to be arbitrary integers.

Theorem 4.1 *For all integers n and all nonnegative integers k,*

$$\binom{n}{k} = \lim_{\epsilon\to 0} \frac{\Gamma(n+1+\epsilon)}{\Gamma(k+1+\epsilon)\Gamma(n-k+1+\epsilon)}. \tag{1}$$

We will adopt Equation 1 as the definition of the binomial coefficients even when k is negative.

Table 4.1 (Binomial Coefficients, $\binom{n}{k}$)

$n\backslash k$	-4	-3	-2	-1	0	1	2	3	4	5	6
6	0	0	0	0	1	6	15	20	15	6	1
5	0	0	0	0	1	5	10	10	5	1	0
4	0	0	0	0	1	4	6	4	1	0	0
3	0	0	0	0	1	3	3	1	0	0	0
2	0	0	0	0	1	2	1	0	0	0	0
1	0	0	0	0	1	1	0	0	0	0	0
0	0	0	0	0	1	0	0	0	0	0	0
-1	-1	1	-1	1	1	-1	1	-1	1	-1	1
-2	3	-2	1	0	1	-2	3	-4	5	-6	7
-3	-3	1	0	0	1	-3	6	-10	15	-21	28
-4	1	0	0	0	1	-4	10	-20	35	-56	84
-5	0	0	0	0	1	-5	15	-35	70	-126	210

Proposition 4.1 (The Six Regions) *Let n and k be integers. Depending on what region of the Cartesian plane (n,k) is in, the following formulas apply:*

1. *If* $n \geq k \geq 0$, *then* $\binom{n}{k} = \binom{n}{k}$.
2. *If* $k \geq 0 > n$, *then* $\binom{n}{k} = (-1)^k \binom{-n+k-1}{k}$.
3. *If* $0 > n \geq k$, *then* $\binom{n}{k} = (-1)^{n+k} \binom{-k-1}{n-k}$
4. *If* $k > n \geq 0$, $\binom{n}{k} = 0$.
5. *If* $0 > k > n$, $\binom{n}{k} = 0$.
6. *If* $k \geq 0 > n$, $\binom{n}{k} = 0$.□

Note that in regions 4–6 there is an extra factor of ϵ in the numerator of the limit, so we are left with zero. Next, region 1 is the classical case, so we have the usual binomial coefficients.

Most of the usual properties of binomial coefficients hold true in all six regions.

Proposition 4.2 (Complementation) *For all integers* n *and* m, $\binom{n}{m} = \binom{n}{n-m}$.□

Proposition 4.3 (Iteration) *For all integers* i, j, *and* k,

$$\binom{i}{j}\binom{j}{k} = \binom{i}{k}\binom{i-j}{j-k}.\square$$

Proposition 4.4 (Pascal) *Let* n *and* k *be integers not both zero, then*

$$\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1}.\square$$

Nevertheless, $\left\lfloor {0 \atop 0} \right\rceil = 1$ while $\left\lfloor {-1 \atop -1} \right\rceil + \left\lfloor {-1 \atop 0} \right\rceil = 1 + 1 = 2$.

Moreover, it is possible to prove a generalization of the binomial theorem via Gamma-Coefficients:

Proposition 4.5 (Gamma Theorem) *For all integers* n,

1. *The coefficients of the formal power series* $(x+1)^n$ *are given by* $[x^k](x+1)^n = \binom{n}{k}$ *where* k *is a nonnegative integer.*
2. *The coefficients of the inverse power series* $(x+1)^n$ *are given by*

$$[x^k](x+1)^n = \binom{n}{n-k} = \binom{n}{k}$$

where k *is a negative integer.*

Proof: Proposition 4.4 and induction.□

5 Inclusion of Hybrid Sets

These new binomial coefficient $\binom{n}{k}$ are always integers, but what do they count. In region 1, they count the number of k element subsets of a given n element set. We claim that given a suitable generalization of the notion of a subset, this is true in general (up to sign[1]).

We define a partial order on these sets which is a generalization of the usual ordering of classical sets and multisets by inclusion; we will give the definition twice: First, an informal motivation and then a formal definition.

Informally, we say f is a subset of g (and write $f \subseteq g$) if one can remove elements one at a time from g (never removing an element that is not a member of g) and thus either acheive f or have removed f. For example, we might start with the hybrid set $f = \{a, b, c, c|d, e\}$. We will remove a few of its elements one at a time. Suppose we remove b, this leaves $\{a, c, c|d, e\}$. Now, b is no longer an element so we can not remove it again. Instead, we might remove d leaving $\{a, c, c|d, d, e\}$. Obviously, we can remove d as many times as we chose. Finally, suppose we remove c leaving $\{a, c|d, d, e\}$. Hence, we have proven two things. Since we were able to remove $\{b, d, c|\}$, we know that $\{b, d, c|\} \subseteq f$. Also, since we were left with $\{a, c|d, d, e\}$, we know that $\{a, c|d, d, e\} \subseteq f$.

Definition 5.1 (Subsets) *Let f and g be hybrid sets. We say that f is a* subset *of g and that g* contains *f and we write $f \subseteq g$ if either $f(x) \ll g(x)$ for all $x \in U$, or $g(x) - f(x) \ll g(x)$ for all $x \in U$ where $\ll$ is a partial ordering of the integers defined as follows: $i \ll j$ if and only if $i \leq j$ and either $i < 0$ or $j \geq 0$.*

Note that the Hasse diagram of the $\ll$ relation is obtained from the usual Hasse diagram of the integers by disconnecting 0 from -1; it consists of the disjoint union of an ascending chain (the nonnegative integers), and a descending chain (the negative integers).

Proposition 5.1 *The ordering defined above is a well defined partial order.*

Proof: Transitivity is the only property worth checking. Suppose $f \subseteq g$ and $g \subseteq h$. We must prove that $f \subseteq h$ in each of the following three cases. Note first that by the above remarks $\ll$ is a partial order.

1. Suppose $f \ll g$ and $g \ll h$. Since $\ll$ is a partial order, $f \ll h$.
2. Suppose $f \subseteq g$ and $h - g \ll h$. Then g is a legal set of removals from h. g must be an ordinary multiset, so by Theorem 5.1, f is a smaller multiset, and must thereof also be a legal set of removals. Hence $f \subseteq h$.

[1] The sign is given by

$$e_k(V)\Big|_{\substack{x=1\\ \forall x \in V}} = \binom{n}{k}.$$

.

3. Suppose $g - f \ll g$ and $g \ll h$. Now, f is a legal set of removals from g. However, g is the remainder of h after another series of removals, so f is a legal set of removals from h.$\square$

Note however that this ordering does not form a lattice. For example, hybrid sets $f = \{|a, b\}$ and $g = \{a|b\}$ have lower bounds $\{a|\}$, $\{a, b|\}$, and $\{a, b, b|\}$, but no greatest lower bound. Conversely, f and g possess no upper bounds.

Theorem 5.1 *The subsets of a classical set f correspond to the classical subsets of a classical set. The subsets of a multiset f correspond to the classical submultisets of a multiset.*

Proof: To construct a conventional subset of a set S, we merely remove some elements subject to the conditions that we only remove elements of S and we don't remove an element twice. The order of removal is not relevant.

To construct a conventional submultiset of a multiset M, we merely remove some elements subject to the conditions that we do not remove any element more times than its multiplicity in M.$\square$

Now, we have the desired interpretation of the binomial coefficients.

Theorem 5.2 *Let n and k be arbitrary integers. Let f be an n-element new set. Then $\binom{n}{k}$ counts the number of k-element hybrid sets which are subsets of f*$\square$

Let us consider this result in each of the six regions of Proposition 4.1.

1. This is the only classical case. In this region, one might count the number of 2 elements subsets of the set $\{a, b, c, d|\}$. By Theorem 5.1, we enumerate the usual subsets: $\{a, b|\}, \{a, c|\}, \{a, d|\}, \{b, c|\}, \{b, d|\}$, and $\{c, d|\}$ but no others.

2. In this region, one might count the number of 2 element subsets of $f = \{|a, b, c\}$. These subsets correspond to what we can remove from f, since what we would have left over after a removal would necessarily contain a negative number of elements. We can remove any of the three elements any number of times, so we have: $\{a, a|\}, \{b, b|\}, \{c, c|\}, \{a, b|\}, \{b, c|\}$, and $\{a, c|\}$.

3. Here we are interested in -5 element subsets of f. Since f contains -3 elements, we must start with f and remove 2 elements. Thus, there is on subset here for each subset in the corresponding position in region 2. In this case they are: $\{|a, a, a, b, c\}, \{|a, b, b, b, c\}, \{|a, b, c, c, c\}, \{|a, a, b, b, c\}, \{|a, b, b, c, c\}$, and $\{|a, a, b, c, c\}$.

4. By Theorem 5.1, there are no 6 elements subsets of the set $\{a, b, c, d|\}$. Once you remove 4 elements, you can not remove anymore.

5. There are no -2 element subset of a -3 element set f. If we remove elements from f, we are left with less than -3 elements, and have removed a positive number of elements. In neither case have we qualified a -2 element hybrid set to be a subset of f.

6. Again by Theorem 5.1, there are no -2 element submits of the set $\{a, b, c, d|\}$, since you are not allowed to introduce elements with a negative multiplicity.

6 Sums and Products with Limits

One often uses the notation $\sum_{n=1}^{i} A(n)$ where n is a nonnegative integer to denote the sum of $A(n)$ over the set $\{1, 2, \ldots, n\}$. In analogy to the situation with integrals, we will define the sum or product of a quantity from an arbitrary integer to another.

Definition 6.1 (Sums and Products with Limits) *We recursively define the sum or product of $A(n)$ from n equals i to j*

$$\sum_{n=i}^{j} A(n) \qquad \textit{or} \qquad \prod_{n=i}^{j} A(n)$$

by two conditions. First,

$$\sum_{n=i}^{i} A(n) = 0 \qquad \textit{and} \qquad \prod_{n=i}^{i} A(n) = 1,$$

and second,

$$\sum_{n=i}^{j+1} A(n) = A(j+1) + \sum_{n=i}^{j} A(n) \qquad \textit{and} \qquad \prod_{n=i}^{j+1} A(n) = A(j+1) \prod_{n=i}^{j} A(n).$$

Just as with integrals, the sum is shift invariant, the sum over an empty interval is zero, and the sum over a positive expression over a "backwards interval" $j < i$ is negative.

The sum or product from i to j can be thought of as the sum or product over the set $\{i..j\}$ where

$$\{i..j\} = \begin{cases} \{i, i+1, \ldots, j-1, j|\} & \text{if } i < j, \\ \emptyset & \text{if } i = j, \text{ and} \\ \{|j+1, j+2, \ldots, i-1\} & \text{if } i > j. \end{cases}$$

For example, $\sum_{i=1}^{4} i = 1 + 2 + 3 + 4 = 10$, and $\sum_{i=1}^{-4} = -(-3) - (-2) - (-1) - 0 = 6$. In general, $\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$ for n regardless of sign. In fact,

Proposition 6.1 *Let $p(x)$ be any polynomial, then there is a polynomial $q(x)$ such that $q(n) = \sum_{i=1}^{n} p(i)$ for all n regardless of sign.*$\square$

Proof: Classical proof by induction can be applied here *mutatis mutandus.*$\square$

As an application of this notion of an interval, and the duality between the elementary and complete symmetric functions mentioned earlier, consider the Stirling numbers of the first kind.

Definition 6.2 (Stirling Numbers of the First Kind) *Let n be an integer, and let k be a nonnegative integer. Then the* Stirling number *of the first kind of degreee n and order k $s(n,k)$ is defined to be the coefficient of x^k in the Taylor series expansion of $(x)_n = \prod_{i=0}^{n-1}(x-i)$.*

Then we have the following beautiful synthesis

Theorem 6.1 *For all integers n (positive or negative) and nonnegative integers k,*

$$s(n,k) = \lim_{\epsilon \to 0} \Gamma(1 - n + \epsilon)^{-1} e_k \left(\frac{1}{\epsilon}, \frac{1}{-1+\epsilon}, \frac{1}{-2+\epsilon}, \ldots, \frac{1}{-n+\epsilon} \right). \square$$

7 Linear Partitions

Just as one can more carefully study the combinatorial properties of the binomial coefficients by introducing the Gaussian coefficient, here we study the generalized Gaussian coefficient. Classically, one defines the *Gaussian coefficient*

$$\binom{n}{k}_q = \frac{(q^n-1)(q^{n-1}-1)\cdots(q^{n+1-k}-1)}{(q^k-1)(q^{k-1}-1)\cdots(q-1)}.$$

We generalize the Gaussian coefficient as follows

Definition 7.1 (Gausssian Coefficient) *Let n be an integer and k be a nonnegative integer. Then define the Gaussian coefficient*

$$\binom{n}{k}_q = \frac{\prod_{i=1}^{k} q^{n+1-i}-1}{(q^k-1)(q^{k-1}-1)\cdots(q-1)}.$$

We will interpret these Gaussian coefficients in terms of *linear partitions*. One usually defines a (linear) partition to be a finite nonincreasing sequence of nonnegative integers. Since we are now giving negative integers a status equal to that of nonnegative ones, this definition is inappropriate.

Definition 7.2 (New Partition) *If λ is a finite sequence of integers of length k such that $\lambda_i \gg \lambda_{i+1}$ for all $1 \le i < k$, then we say that λ is a new partition. The λ_i are called the parts of λ, and we define $|\lambda|$ to be the sum of the parts of λ. We say that λ is a partition of $|\lambda|$*

For example, $(5,3,2,2,0)$ and $(-3,-7,-7,-32)$ are new partitions, but $(3,1,0,-1)$ is not.

It is well known (see for example Knuth) that for n and k nonnegative, $\binom{n}{k}_q$ is a monic polynomial of degree nk. Its coefficient of q^t is the number of partitions λ of t of length k with all parts less than or equal to n. In general,

Theorem 7.1 *Let k be a nonnegative integer, and n and t be arbitrary integers. Then $\binom{n}{k}_q = \sum_t c_t q^t$ where c_t is the number of partitions λ of t of length k with all parts $\ll n$.*$\square$

8 Functions

We now define the hybrid sets of "functions" and "injections" between a classical set S and an arbitrary hybrid set A.

Definition 8.1 (Functions) *Given a classical set S, and a hybrid set A. Define $Fun(S,A)$ and $Mono(S,A)$ to be hybrid sets of functions from S to the universe U. The multiplicity of $f : S \to U$ in $Fun(S,A)$ is $\prod_{x\in S} A(f(x))$. Whereas, the multiplicity of f in $Mono(S,A)$ is $\prod_{x\in U}\binom{A(x)}{\#f^{-1}(x)}$ where $\#f^{-1}(x)$ is the size of the inverse image of x under f $|f^{-1}(x)|$.*

For example, if A is a classical set then $Fun(S,A)$ is the set of functions from S to A, and $Mono(S,A)$ is the set of such injections.

Here we see that $\#Fun(S,A) = \#A^{(\#S)}$, and confirm Stanley's formula that $\Omega(P,-n)$ counts poset homomorphism onto chains, if $\Omega(P,n)$ counts strict poset homomorphisms onto chains.

9 Other Applications

Since submitting this article, the author has read about surreal numbers [1] which can be interpretted as an example of this theory.

Another application (concerning the connection constants between polynomial sequences and/or inverse formal power series sequences) will be appear in a separate article. [2]

References

[1] J. W. Conway, "On Numbers and Games," Academic Press, London, 1976.

[2] E. Damiani, O. D'Antona, and D. Loeb, *The Complimentary Symmetric Function: Connection Constants Using Negative Sets,* To appear.

[3] D. Knuth, *Subspaces, Subsets, and Partitions,* Journal of Combinatorial Theory **10** (1971) 178–180.

[4] D. Loeb, *A Generalization of the Binomial Coefficients*, SIAM Journal of Discrete Mathematics, To appear.

[5] D. Loeb, *A Generalization of the Stirling Numbers*, SIAM Journal of Discrete Mathematics, To appear.

[6] D. Loeb and G.-C. Rota, *Formal Power Series of Logarithmic Type*, Advances in Mathematics, **75** (1989), 1–118.

[7] I. G. Macdonald, "Symmetric Functions and Hall Polynomials," *Oxford Mathematical Monographs*, Claredon Press, Oxford, 1979.

[8] S. Roman, *A Generalization of the Binomial Coefficients*, To Appear.