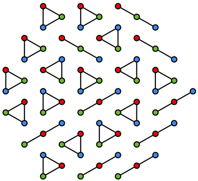
Trimer covers in the triangular grid

James Propp, UMass Lowell
Open Problems in Algebraic Combinatorics
May 18, 2022



Slides for this talk are at
http://faculty.uml.edu/jpropp/opac22.pdf

A video of the talk is available through
http://www.samuelfhopkins.com/OPAC/opac.html

Earlier related talks

November 30, 2021, Conjectural Enumerations of Trimer Covers of Finite Subgraphs of the Triangular Lattice. Video: https://youtu.be/ANzukenVxvU

March 30, 2022, Tiling problems, old and new. Video: https://faculty.uml.edu/jpropp/rutgers22.mp4

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Example: downsets, upsets, independent sets $\to$ rowmotion

Example: ASMs, DPPs, and TSSCPPs

When math hurts

David Robbins: "These conjectures are of such compelling simplicity that it is hard to understand how any mathematician can bear the pain of living without understanding why they are true."

(Cf. Walter Bagehot: "One of the greatest pains to human nature is the pain of a new idea.")

My goal today is to give you several (hopefully not too uncomfortable) itches, with many open problems, and two half-open problems (one half-open from the left and the other half-open from the right).

Enumerating dimer covers

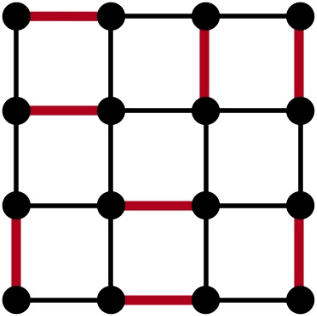
A dimer in a graph $G = (V, E)$ is just an edge $e \in E$.

A dimer cover of a graph (V, E) (aka a perfect matching) is a set $E' \subseteq E$ of edges with the property that each $v \in V$ belongs to exactly one $e \in E'$.

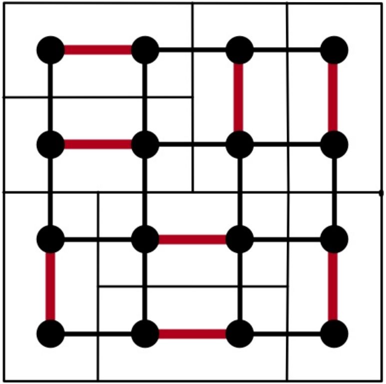
Main problem: How many dimer covers does G have?

Dimer problems are dual to tiling problems.

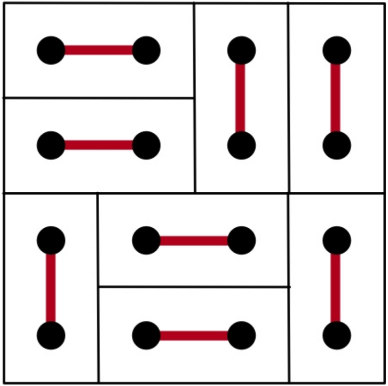
From



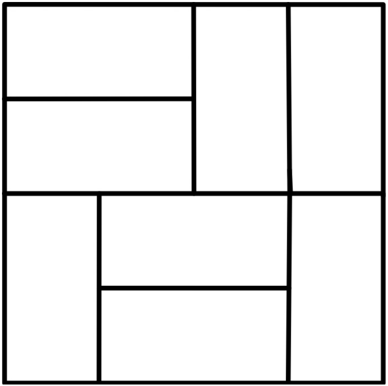
From dimers



From dimers to



From dimers to dominos



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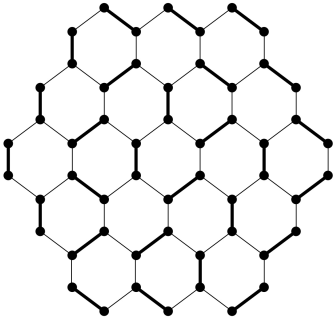

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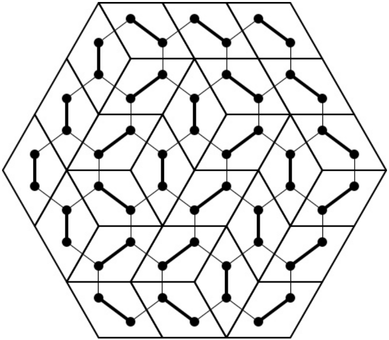
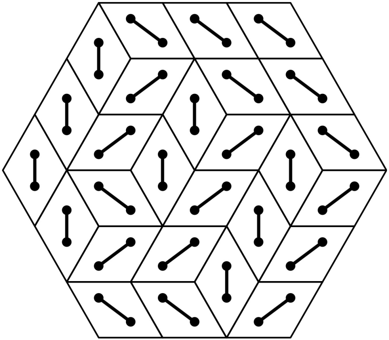
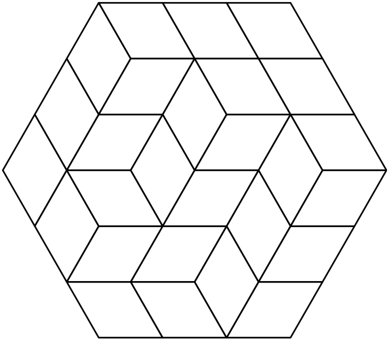


From dimers to



From dimers to lozenges



Counting dimer covers

Temperley-Fisher and Kasteleyn (1961): The number of domino tilings of a $2n$-by-$2n$ square is

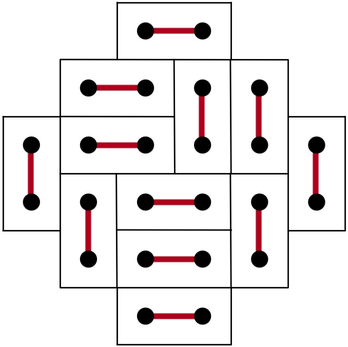
$$\prod_{j=1}^{n}\prod_{k=1}^{n}(4\cos^2\frac{j\pi}{2n+1}+4\cos^2\frac{k\pi}{2n+1})$$

MacMahon ($\sim$1896), and MacDonald (1967?): The number of lozenge tilings of a regular hexagon of side-length n is

$$\prod_{i=1}^{n}\prod_{j=1}^{n}\prod_{k=1}^{n}\frac{i+j+k-1}{i+j+k-2}$$

(exponential in n^2; largest prime factor is less than $3n$).

Aztec diamonds



Elkies-Kuperberg-Larsen-Propp (1992): The Aztec diamond graph of order n has exactly $2^{n(n+1)/2}$ dimer covers.

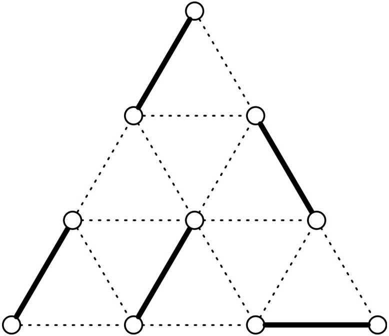
The permanent-determinant method

Temperley and Fisher, and independently Kasteleyn, gave a general method for counting perfect matchings of planar graphs.

They showed that the number of perfect matchings of a planar bipartite graph with n nodes of each color is equal to the square root of the determinant of a modified version of the n-by-n adjacency matrix of the graph in which some of the 1's get flipped to -1's (or more generally get replaced by roots of unity).

The Pfaffian-Hafnian method

When the planar graph is not bipartite, there's still an exact formula using Pfaffians rather than determinants.



Big question

Is there a general formula for counting trimer covers of (finite) planar graphs?

I don't know.

Are there nice formulas counting trimer covers of some particular graphs?

Apparently!

Let's restrict attention to tripartite graphs, and require a trimer to contain one vertex in each color class.

Conway, Lagarias, and Thurston

Part of what originally got me interested in tilings thirty years ago was an article by Conway and Lagarias along with a follow-up article by Thurston, posing and solving tiling existence problems that couldn't be solved using earlier methods.

Tiling with Polyominoes and Combinatorial Group Theory

J. H. Conway

Princeton University, Princeton, New Jersey

AND

J. C. Lagarias

AT&T Bell Laboratories, Murray Hill, New Jersey

Communicated by Andrew Odlyzko

Received May 3, 1988

When can a given finite region consisting of cells in a regular lattice (triangular, square, or hexagonal) in $\mathbb{R}^2$ be perfectly tiled by tiles drawn from a finite set of tile shapes? This paper gives necessary conditions for the existence of such tilings using *boundary invariants*, which are combinatorial group-theoretic invariants associated

Conway's Tiling Groups

William P. Thurston, *Princeton University*

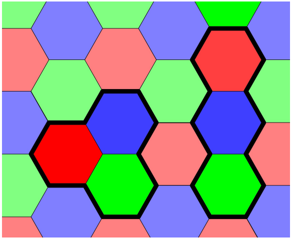
Dr. Thurston's numerous distinctions and honors include the Oswald Veblen Prize in Geometry, the Alan T. Waterman Award of the NSF, and the Fields Medal. His Ph.D. is from Berkeley, in 1972.

1. Introduction

John Conway discovered a technique using infinite, finitely presented groups that in a number of interesting cases resolves the question of whether a region in the plane can be tessellated by given tiles. The idea is that the tiles can be interpreted as describing relators in a group, in such a way that the plane region can be tiled, only if the group element which describes the boundary of the region is the trivial element 1.

Trimers and trihexes

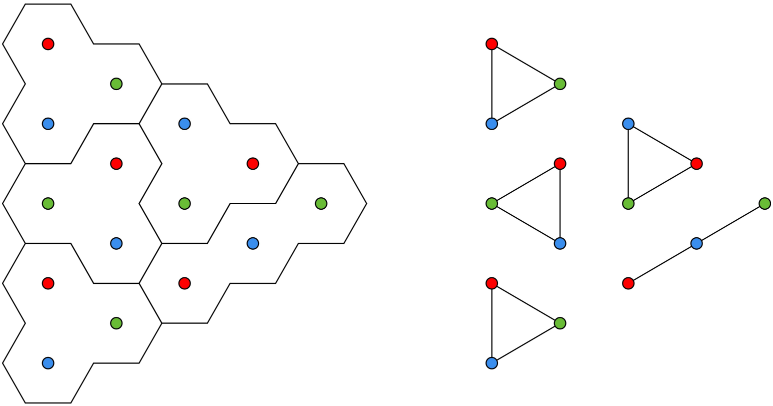
Conway et al. studied tilings in the hexagonal grid, using tiles ("trihexes") composed of three hexagons, with each tile containing one hexagon from each of the 3 color-classes under the natural way of 3-coloring the hexagonal grid.



I call these two shapes of tiles stones and bones respectively.

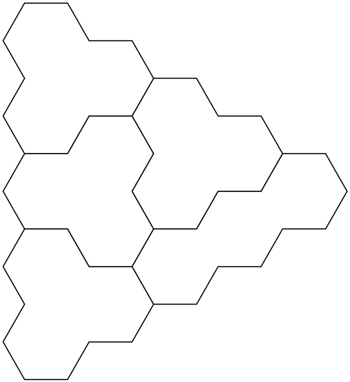
Trihexes and trimers

Of course tilings of this kind are dual to trimer covers.



Trimers in a triangle

It's not hard to show that can tile a honeycomb triangle with n hexagons on a side (denoted by T_n) using stones and bones if and only if n is congruent to 0 or 2 (mod 3).



Enumeration?

However, if we count the tilings as a function of n, the numbers we get don't suggest a simple formula. Leaving out the 0 terms we have:

1, 3, 30, 246, 25321, 591103, 603105309, 41333676318, 410382321560202, 83918368144461643, 8025244898075570226296, 4941312847984149589980261, ...

(These numbers were computed using a program written by David desJardins.)

Forbidding tiles

What was amazing about the two papers is that they used combinatorial group theory and/or combinatorial topology to analyze what happens when you allow only stones, or only bones.

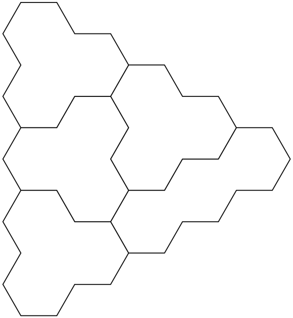
Theorem: T_n can be tiled by stones if and only if $n \equiv 0, 2, 9,$ or 11 (mod 12).

Theorem: T_n can't be tiled by bones for any $n > 0$.

The main tool used to prove these theorems was a novel invariant.

The Conway-Lagarias invariant

For a simply-connected honeycomb region R and a tiling T of R by stones and bones, let $I(T)$ be the number of rightward-pointing stones minus the number of leftward-pointing stones. E.g., $I(T) = 3 - 1 = 2$ for the tiling T shown below.



The Conway-Lagarias invariant

Conway and Lagarias showed that $I(T)$ depends only on the region R, not the tiling T!

(In particular, if a region R has a stones-and-bones tiling T in which the number of rightward-pointing stones and the number of leftward-pointing stones aren't equal, then the Conway-Lagarias invariant of R is nonzero, so the region R can't be tiled with bones alone. This is how one shows that T_n can never be tiled by bones.)

Stone tilings

When n is congruent to 0, 2, 9, or 11 (mod 12), one can count stone tilings of T_n.

Unfortunately this sequence does not seem to be governed by a nice formula either. Leaving out the 0 terms we have:

1, 1, 2, 8, 12, 72, 185328, 4736520, 21617456, 912370744, 3688972842502560, 717591590174000896, 9771553571471569856, 3177501183165726091520, ...

What to do?

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"Try Domain Wall Boundary Conditions™!"

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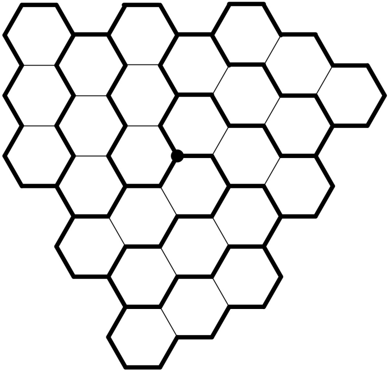
Benzels are to honeycomb triangles as Aztec diamonds are to squares; they're just-barely-tileable at the boundary so that large-scale structures are likely to propagate into the interior of the region if you choose a state uniformly at random.

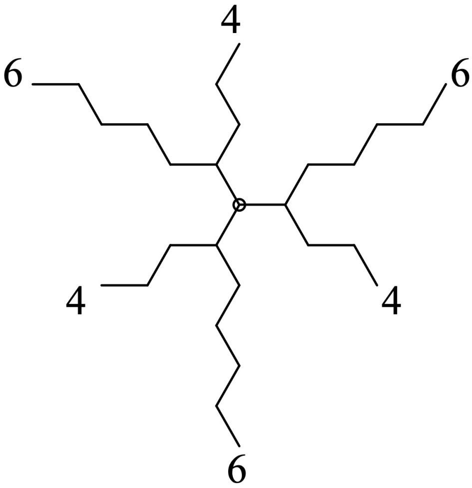
For motivation, watch the video of my 2021 talk at the Combinatorics and Arithmetic for Physics workshop.

What do benzels look like?

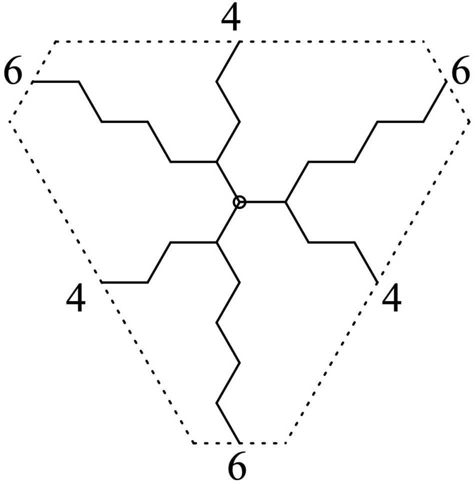
Benzels form a two-parameter family, with a and b satisfying $a \leq 2b - 2$ and $b \leq 2a - 2$. Here is the 5,7-benzel:



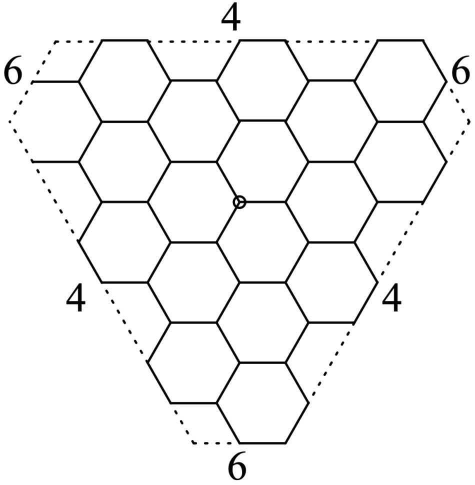
Building a benzel, step 1



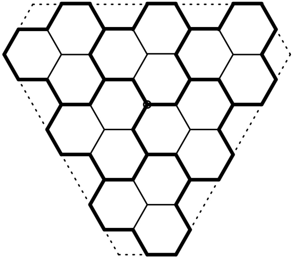
Building a benzel, step 2



Building a benzel, step 3



The 4,6-benzel, tiled by stones



All the stones point the same way, so the Conway-Lagarias invariant tells us it has only one tiling.

I believe (but have not yet proved) that the a, b benzel can be tiled by stones alone exactly when $a + b$ is 1 (mod 3).

Tiling with bones alone: audience quiz

On the other hand, hardly any benzels can be tiled by bones alone. For instance, with $a, b \leq 10$, only the 5,7-benzel (and the 7,5-benzel) can be tiled by bones alone.

But there is another (with larger a, b). Any guesses?

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So can the 12,15-benzel.

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So can the 12,15-benzel.

But there is another. Any guesses?

So can the 22,26-benzel.

What's the pattern?

Paired pentagonal numbers

Theorem (March 2022): The a, b-benzel can be tiled by bones alone only if a, b are paired pentagonal numbers $k(3k \pm 1)/2$.

Proof sketch:

1. Compute the Conway-Lagarias invariant $I(R)$ (three cases)
2. Show that the only feasible case is $a + b \equiv 0 \pmod 3$, with $I(R) = (a + b - 3a^2 + 6ab - 3b^2)/6$.
3. Set $I(R) = 0$. Solving for b in terms of a, show that $24b + 1$ must be a perfect square.
4. Show that this happens only when a and b are paired pentagonal numbers.

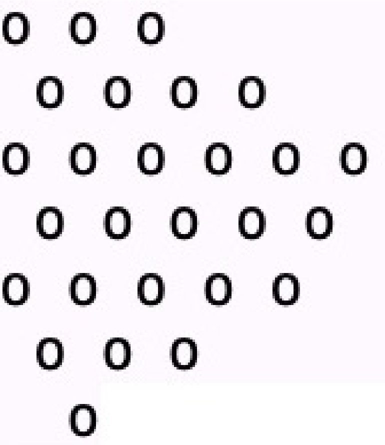
Is the problem half-open or half-closed?

It remains to show that when a and b are of this form, at least one tiling exists. That's an open problem!

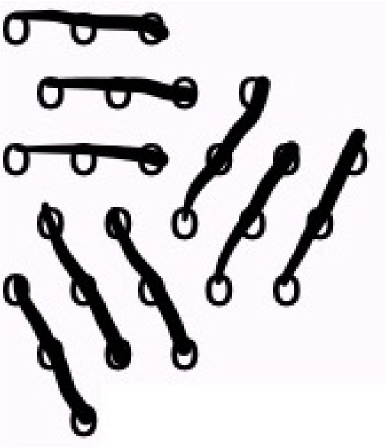
There appear to be lots of such tilings, but I don't know of any systematic way to construct them, so I don't know how to prove that they exist for all k.

I've prepared some sheets with the nodes of a 12,15-benzel graph on one side and the nodes of a 22,26-benzel on the other. Your task is to find a trimer cover on each graph. Can you find a systematic way that works for both graphs?

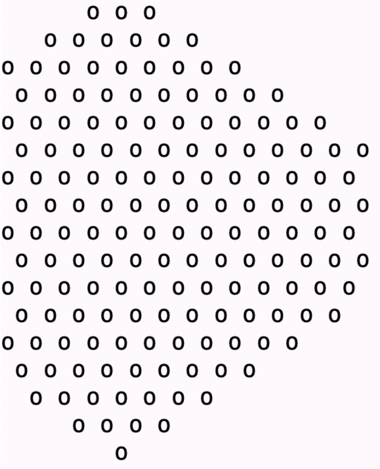
In other words, turn something like this...



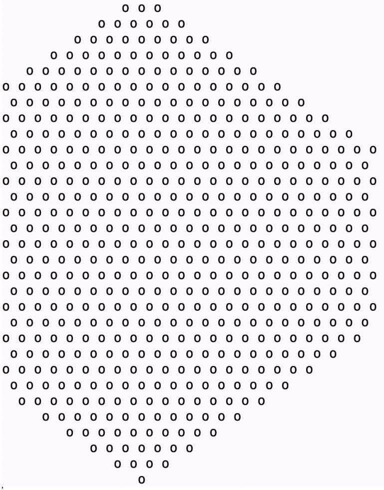
... into something like this:



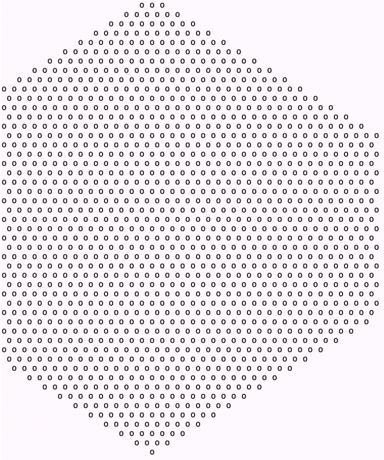
The 12,15-benzel



The 22,26-benzel



The 35,40-benzel



Alternative approach

Can you prove that every benzel has a stones-and-bones tiling in which all the stones point the same way?

1 kind of stone, 2 kinds of bones

Allow 1 (of the 2) kinds of stones and 2 (of the 3) kinds of bones. How many tilings are there? With $3 \leq a, b \leq 15$:

2	0	0	0	0	0	0	0	0	0	0	0	0
0	0	2	0	0	0	0	0	0	0	0	0	0
0	2	0	0	0	0	0	0	0	0	0	0	0
0	0	0	8	0	0	0	0	0	0	0	0	0
0	0	0	0	0	8	0	0	0	0	0	0	0
0	0	0	0	8	0	0	0	0	0	0	0	0
0	0	0	0	0	0	48	0	0	0	0	0	0
0	0	0	0	0	0	0	0	48	0	0	0	0
0	0	0	0	0	0	0	48	0	0	0	0	0
0	0	0	0	0	0	0	0	0	384	0	0	0
0	0	0	0	0	0	0	0	0	0	0	384	0
0	0	0	0	0	0	0	0	0	0	384	0	0
0	0	0	0	0	0	0	0	0	0	0	0	3840

Double factorials

Conjecture: When a and b both equal $3n$, or when one is $3n+1$ and the other is $3n+2$, the number of tilings of the a, b-benzel with 2 kinds of bones and 1 kind of stone allowed is $2 \times 4 \times \cdots \times 2n$.

UPDATE: This problem is now only half-open. Ben Young has found an explicit description of $2 \times 4 \times \cdots \times 2n$ tilings, so all that remains to do is prove that there aren't any others.

2 kinds of stones, 2 kinds of bones

Allow 2 (of the 2) kinds of stones and 2 (of the 3) kinds of bones. How many tilings are there? With $3 \leq a, b \leq 10$:

2	1	1	1	1	1	1	1
1	4	6	1	1	1	1	1
1	6	1	16	22	1	1	1
1	1	16	48	1	68	90	1
1	1	22	1	224	512	1	304
1	1	1	68	512	1	3360	6736
1	1	1	90	1	3360	15360	1
1	1	1	1	304	6736	1	168960

Hiding along one nearly extremal diagonal are 2, 6, 22, 90, 394, 1806, 8558, 41586, . . . ; these are the large Schroder numbers (A006318). Hiding along a parallel diagonal are 1, 4, 16, 68, 304, 1412, 6752, 33028, . . . (A006319).

2 kinds of stones, 2 kinds of bones

The really big numbers in this table lie near the main diagonal. Four subsequences parallel to the main diagonal

1,16,3360,9371648,347950546944, . . .

2,48,15360,65601536,3737426853888, . . .

4,224,168960,1705639936,229940737867776, . . .

6,512,591360,9160359936,1897011087409152, . . .

have the property that the nth term grows exponentially in n^2, but the largest prime factor of the nth term is less than $4n$.

2 kinds of stones, 2 kinds of bones

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Four subsequences parallel to the main diagonal

1,16,3360,9371648,347950546944, ...

2,48,15360,65601536,3737426853888, ...

4,224,168960,1705639936,229940737867776, ...

6,512,591360,9160359936,1897011087409152, ...

have the property that the nth term grows exponentially in n^2, but the largest prime factor of the nth term is less than $4n$.

$\Rightarrow$ Product formulas!
David desJardins figured out one of the product formulas; to figure out the other three, we'll need more terms.

2 kinds of stones, 3 kinds of bones

There are no conjectural product formulas, but two diagonals of the table have regular 2-adic behavior.

With $a = n$ and $b = 2n - 3$, the number of tilings goes
$0,0,0,0,\ldots \pmod 2$;
$2,2,2,2,\ldots \pmod 4$;
$2,6,2,6,\ldots \pmod 8$.

I conjecture that the residues mod 2^k are periodic mod 2^j for some j ("2-adic continuity").

Likewise for $a = n$ and $b = 2n - 4$.

Where does 2-adic continuity come from?

I know of just one (proved) example of 2-adic continuity arising from enumeration of tilings.

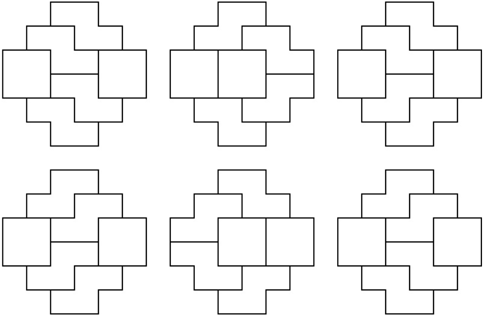
Let $T(n)$ be the number of domino tilings of a $2n$-by-$2n$ square. $T(n)$ can be expressed as $2^n S(n)^2$, where $S(n)$ counts the domino tilings of the nth Pachter half-square.

Henry Cohn showed that the sequence $S(n)$ (https://oeis.org/A065072) is 2-adically continuous.

His proof used the exact product formula, but for trimer covers we have no exact formula!

2-adic properties for tetramer covers

I have also observed 2-adic properties for tetramer covers of Aztec diamond graphs: see my preprint "Some 2-adic conjectures concerning polyomino tilings of Aztec diamonds".



Honeycomb triangles revisited

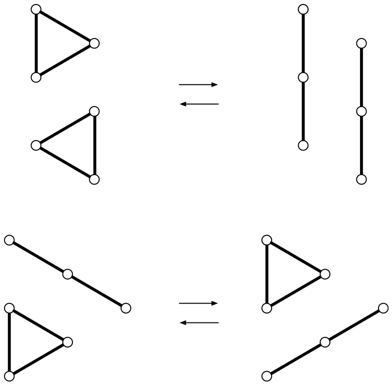
Let's take another look at the sequence https://oeis.org/A334875 counting tilings of honeycomb triangles by stones.

The multiplicity of the prime 2 in the factorizations of the nonzero terms in this sequence are 0, 0, 1, 3, 2, 3, 4, 3, 4, 3, 5, 8, 6, 8 which seem to show an upward drift.

Conjecture: The number of tilings goes to zero 2-adically.

Mutations

Open problem: Can every stones-and-bones tiling T of a simply-connected honeycomb region be obtained from every other tiling T' by means of a sequence of "2-flips", each of which replaces 2 tiles by 2 other tiles?



Mutations

If true, this would yield the invariance of the Conway-Lagarias invariant as a corollary, since it can be checked that 2-flips don't change the value of $I(T)$.

Scott Sheffield has shown that if we reduce the tile set by forbidding bones oriented in a particular direction, the claim becomes true.

Random tilings

Preliminary results indicate that along the boundary of a benzel, a random stones-and-bones tiling is “close to nonrandom”.

I would expect that the tilings exhibit long-range order.

But: how do you generate trihex tilings uniformly at random? I don’t know a smart way, but there is a brute force way that would still be informative.

Needed: coders!

Ideas for proving the conjectures?

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Hyperdeterminants?

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R-matrices?

Hyperdeterminants?

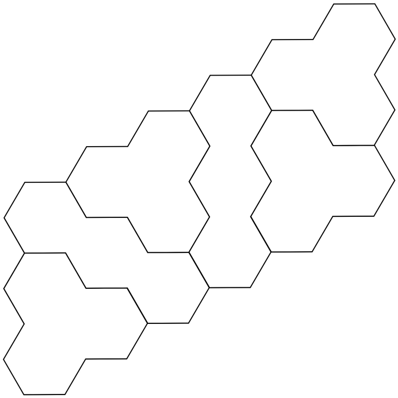
How might you invent matrices and determinants if all you knew about was dimers?

An accessible counting problem

Form a honeycomb parallelogram with 3 hexagons on its short sides and n hexagons on its long sides.

Forbid bones that are parallel to the long sides of the parallelogram, but allow the other four types of tiles.

An accessible counting problem



How many tilings are there?

Thanks!

That's all I got; thank you for listening!

Slides for this talk are at http://jamespropp.org/opac22.pdf

For updates, email me (jamespropp at-sign gmail.com) and ask to subscribe to the DOMINO listserv.