Bridging the gap between the continuous and the discrete

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a talk given in honor of the newly-inducted members
of the UMass Lowell chapter of Pi Mu Epsilon

Math from grade school to college and beyond

Arithmetic

$\Downarrow$

Algebra, Geometry, Trigonometry

$\Downarrow$

Calculus

$\Downarrow$

Differential equations, Discrete mathematics,
Topology, Number theory,
Mathematical logic, Theory of computation,
Dynamical systems, Probability,
Real analysis, Complex analysis, ...

Connecting the continuous and the discrete

PROBLEM 1: If $(Df)(t) = t$ for all real numbers t and $f(1) = 2$, find $f(3)$.

(Note: Df is another way of writing f'.)

PROBLEM 2: Prove that $1 + 2 + \cdots + n = n(n + 1)/2$ for all positive integers n.

Solving a continuous-math problem

To solve Problem 1, we use a basic lemma of calculus that says that **two functions with the same derivative must differ by a constant**.

Since $f(t)$ and $t^2/2$ have the same derivative (namely t), we must have $f(t) = t^2/2 + C$ for some constant C.

How do we determine C?

To evaluate the constant, plug in $t = 1$:

$f(1) = 2$ and $1^2/2 + C = 1/2 + C$,

so $2 = 1/2 + C$ and $C = 3/2$.

Then we get $f(3) = 3^2/2 + C = 9/2 + 3/2 = 12/2 = 6$.

Solving a discrete-math problem

To solve Problem 2, we can use mathematical induction, but I want to show you a different way, using “the other calculus”:

the “calculus of finite differences”, or “difference calculus” (not to be confused with the “differential calculus” of Leibniz and Newton that we teach you in 92.131 and 92.141).

"The Other Calculus"

Just as Df is the function satisfying

$$(Df)(t) = \lim_{h \to 0} \frac{f(t+h) - f(t)}{h},$$

we define Δf to be the function whose value at t is $f(t+1) - f(t)$:

$$(\Delta f)(t) = f(t+1) - f(t).$$

Just as we call Df the **first derivative** of f, we call Δf the **first difference** of f.

From functions to sequences

Note that Δf makes sense even if the function f is defined only when t is an integer (not true for Df!).

And this is good news for us (if we want to apply Δ to Problem 2), because it's not clear what $1 + 2 + \cdots + n$ even means if n isn't an integer!

Given a sequence

$$a = (a_1, a_2, a_3, ...),$$

we define its difference sequence Δa as

$$\Delta a = (a_2 - a_1, a_3 - a_2, ...).$$

Two examples

$$\Delta(1, 3, 5, 7, \dots) = (2, 2, 2, 2, ...)$$

$$\Delta(2, 4, 6, 8, \dots) = (2, 2, 2, 2, ...)$$

A basic lemma

Basic lemma of difference calculus: If two sequences have the same difference sequence, they must differ by a constant.

(Compare this with the analogous property of differential calculus: two functions that have the same derivative must differ by a constant.)

Note that the implication goes the other way too — two sequences that differ by a constant must have the same difference sequence — but that proposition is simple algebra.

To prove the basic lemma of difference calculus, you need to use mathematical induction.

A proof of the basic lemma of difference calculus

Claim: If $a_2 - a_1 = b_2 - b_1$ and $a_3 - a_2 = b_3 - b_2$ and so on, then $a_n - b_n$ is independent of n.

Proof: We have $a_2 - b_2 = a_1 - b_1$, $a_3 - b_3 = a_2 - b_2$, etc.

So $a_1 - b_1 = a_2 - b_2 = a_3 - b_3 = \ldots$.

So by induction $a_n - b_n$ is independent of n, QED.

(Curious fact: You can use the basic lemma of difference calculus to PROVE the principle of mathematical induction!)

We've got a hammer; let's find some nails

Now let's apply the basic lemma of difference calculus to the sequences $a_n = 1 + 2 + ... + n$ and $b_n = n(n+1)/2$.

$$\begin{aligned}(\Delta a)_n &= a_{n+1} - a_n \\ &= (1 + 2 + ... + n + n + 1) - (1 + 2 + ... + n) \\ &= n + 1\end{aligned}$$

$$\begin{aligned}(\Delta b)_n &= b_{n+1} - b_n \\ &= (n+1)(n+2)/2 - n(n+1)/2 \\ &= (n + 2 - n)(n+1)/2 \\ &= n + 1\end{aligned}$$

Since $\Delta a = \Delta b$, the fundamental lemma of difference calculus tells us that $a_n - b_n = C$ for all n, for some constant C.

Finishing the job

We've shown that for $a_n = 1 + 2 + \cdots + n$ and $b_n = n(n+1)/2$ we have $a_n - b_n = C$ for some constant C.

How do we determine C?

Just plug in!

$a_n = 1 + 2 + \cdots + n$, so $a_1 = 1$;

$b_n = n(n+1)/2$, so $b_1 = 1$;

so $C = a_1 - b_1 = 0$.

So $a_n - b_n = C = 0$ for all n.

So $a_n = b_n$ for all n, QED.

The moral

The analogy between differential equations and difference equations is quite deep.

For instance, just as we use linear algebra to solve linear differential equations, we use linear algebra to solve linear difference equations, like the famous Fibonacci difference equation $F_{n+1} = F_n + F_{n-1}$.

(In each case, the basic lemma says that the kernel of a particular operator is 1-dimensional; in one case the operator is D, in the other case it's Δ.)

But even more broadly, the analogy between the discrete world and the continuous world is quite deep (though at times it takes some work to find the tools that bridge the gap).

The real moral

And even more broadly, the seemingly disconnected branches of study that appeared on my first slide are part of one big, living organism called Mathematics.

If you continue to pursue mathematics, you'll often find yourself studying very narrow and arcane issues, but you'll prosper if at frequent intervals you remind yourself of the unity of mathematics, and always strive for a unifying perspective.

Congratulations on your achievements, past and future!