1 Background

For a matrix A, let $A(i,j)$ denote its (i,j)th entry. Suppose A_n is an n-by-n matrix and A_{n+1} is an $(n+1)$-by-$(n+1)$ matrix. We define A_k recursively in terms of A_{k+1} and A_{k+2} by $A_k(i,j)A_{k+2}(i+1,j+1) = A_{k+1}(i,j)A_{k+1}(i+1,j+1) + \lambda A_{k+1}(i+1,j)A_{k+1}(i,j+1)$, and write $A_{n+1} \to A_n \to A_{n-1} \to \cdots \to A_1$. The *$\lambda$-determinant of a pair* (A_n, A_{n+1}) is the sole entry of A_1. Let the *λ-determinant of a matrix* A_n, be the λ-determinant of the pair (A_n, C), where C is the $(n+1)$-by-$(n+1)$ matrix each of whose entries is 1. Note that the (-1)-determinant of a matrix is just its determinant. Let $\Lambda(A)$ denote the 1-determinant of A.

Let the *Aztec diamond graph* be the dual graph of an Aztec diamond, and a *weighted Aztec diamond graph (WAD)* be an Aztec diamond graph with associated edge weights. Designate by $W(F)$ the sum of the weighted perfect matchings of a WAD F. An S-WAD is a WAD whose entries are chosen from the set S; in this paper we will be using $\{0,1\}$-WADs almost exclusively.

2 Kuo Recurrence

Consider the Aztec diamond graph as tilted 45 degrees. Call two faces F, G of the graph *vertex adjacent* (in which case we write $F|G$) if they share a common vertex but not an edge, and *vertex connected* if either $F|G$ or there is a finite set $\{F_1, F_2, \ldots, F_n\}$ of faces in the graph such that $F_i|F_{i+1}$ for all i, $F|F_1$, and $F_n|G$. Call a face a *major face* if it is vertex connected to the upper left face and a *minor face* otherwise. The major faces of an Aztec diamond F form a finite square lattice; call the one in column i, row j $F(i,j)$. Similarly, the minor faces of F form a finite square lattice; label them $\overline{F}(i,j)$. If F is a WAD, let $F(i,j)_{ne}, F(i,j)_{nw}, F(i,j)_{se}, F(i,j)_{sw}$ represent the weightings of the northeast, northwest, southeast, and southwest edges bordering $F(i,j)$, respectively.

Given a WAD F of order n, let $F_{ne}, F_{nw}, F_{se}, F_{sw}$ be the northeast, northwest, southeast, and southwest order $n-1$ weighted Aztec subdiamonds, and let F_m be the inner order $n-2$ weighted Aztec subdiamond. If we let the *edge multiplying factors* n_e, n_w, s_e, s_w be the weights of the northeast, northwest, southeast, and southwest edges of F, then the number of weighted matchings of F is given in [1] by the recurrence

$$W(F)W(F_m) = n_e s_w W(F_{nw})W(F_{se}) + n_w s_e W(F_{ne})W(F_{sw}). \tag{1}$$

2.1 Edge Weights

Given a $\{0,1\}$-WAD F, create a square matrix M whose (i,j)th entry is equal to $F(i,j)_{nw}F(i,j)_{se} + F(i,j)_{ne}F(i,j)_{sw}$ (the *edge factor* of $F(i,j)$). If F is weighted all 1 within some Aztec suboctagon and in a brickwork pattern outside this region, the 1-determinant of M is equal to the number of weighted matchings of F. This comes almost immediately from (1), the only trick being that it works in general only if the possible weights are 0 and 1, in which case the edge multiplying factors drop out.

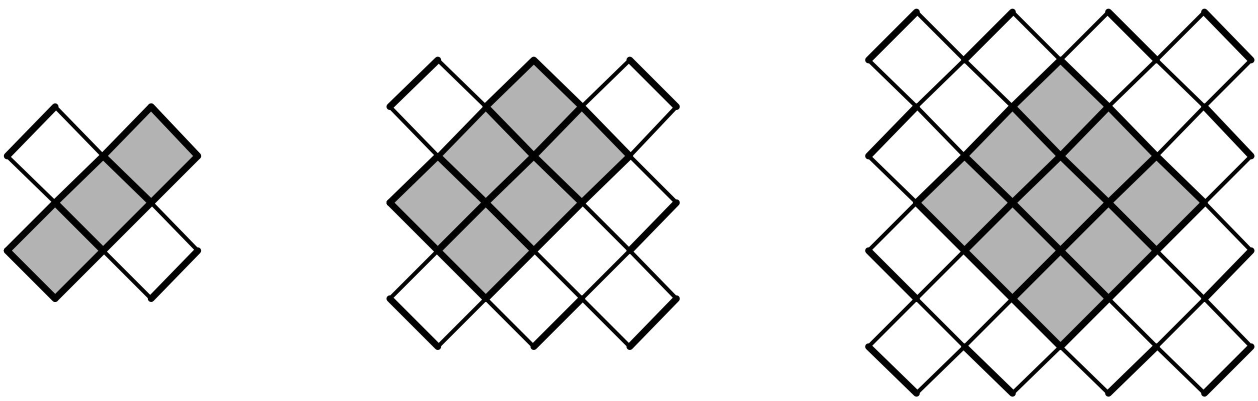
2.2 Face Weights

Given an order n Aztec diamond graph F, the *total minor faces* of F are the canonically-defined $\overline{F}(i, j)$ where i, j range from 0 to $n+1$ (i.e. defined such that they uphold the lattice structure of the minor faces). The *face weight* of a total minor face $\overline{F}(i, j)$ is either (i) the number of edges selected immediately around $\overline{F}(i, j)$ in a nonzero weighted perfect matching of F if this is a fixed value in any such matching, or (ii) 1 otherwise.

Given an order n $\{0, 1\}$-WAD F whose edges are all weighted 1 within some Aztec suboctagon, and the rest of whose edges are weighted in a brickwork pattern, let M be the $(n+1)$-by-$(n+1)$ matrix whose (i, j)th entry is the face weight of $\overline{F}(i, j)$. Then if F is weighted all 1 within some Aztec suboctagon and in a brickwork pattern outside this region, the 1-determinant of M is equal to the number of weighted matchings of F. The proof is immediate, since the intermediate n-by-n matrix used to take the 1-determinant is equal to the matrix used in the edge weighting picture.

3 Embedding Rectangles

The above techniques can be used to find the number of perfect matchings of even-by-even rectangles by "embedding" them in Aztec diamonds via appropriate weighting. All edges of the rectangle are weighted 1, and the rest of the edges of the Aztec diamond are weighted in a brickwork pattern (if the can be; otherwise we must choose a different embedding of the rectangle). In the following examples, an edge weight of 1 is represented by a darkened line, and all other edges are weighted 0. The shading is included only to highlight the region under consideration.



In this figure the leftmost example corresponds to $\begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix} \rightarrow 5$, the middle example to $\begin{bmatrix} 1 & 2 & 1 \\ 2 & 2 & 1 \\ 1 & 1 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 6 & 4 \\ 4 & 1 \end{bmatrix} \rightarrow 11$, and the rightmost example to $\begin{bmatrix} 0 & 1 & 1 & 0 \\ 1 & 2 & 2 & 1 \\ 1 & 2 & 2 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 4 & 1 \\ 4 & 8 & 4 \\ 1 & 4 & 1 \end{bmatrix} \rightarrow \begin{bmatrix} 12 & 12 \\ 12 & 12 \end{bmatrix} \rightarrow 36$.

References

[1] E. Kuo, *Applications of Graphical Condensation for Enumerating Matchings and Tilings*, preprint 2002