Solution to Problem 10877 from the American Mathematical Monthly

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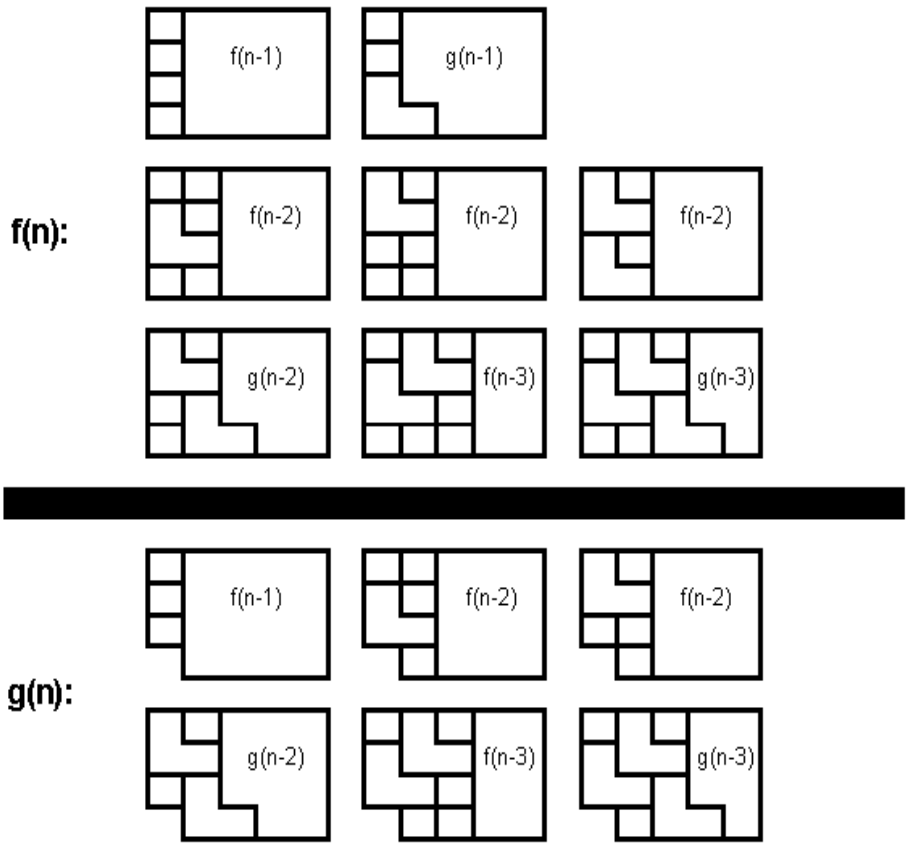
An L-tile is a 2×2 *square with the upper right* 1×1 *square removed; no rotations are allowed. Let* a_n *be the number of tilings of a* $4 \times n$ *rectangle using tiles that are either* 1×1 *squares or L-tiles. Find a closed form for the generating function* $1 + a_1 x + a_2 x^2 + a_3 x^3 + \ldots$.

Let an L-tiling be a tiling that uses only L-tiles with the upper right corner missing) and 1×1 squares.

Let $f(n)$ = the number of L-tilings of a $4 \times n$ rectangle, and $g(n)$ = the number of L-tilings of a $4 \times n$ rectangle with the bottom left corner missing.

We can build L-tilings in such a way that the possible tilings can be categorized according to the appearance of the first 4 columns.

The Possible Ways to Build up an L-Tiling of a 4 x N Rectangle



From the analysis of this pattern, we get the following recurrence relations for $n > 3$:

$$f(n) = f(n-1) + g(n-1) + 3\ f(n-2) + g(n-2) + f(n-3) + g(n-3)$$

$$g(n) = f(n-1) + 2\ f(n-2) + g(n-2) + f(n-3) + g(n-3)$$

These relations also hold for $n = 1,\ 2$, and 3, with the convention $f(0) = 1,\ g(0) = f(-1) = g(-1) = f(-2) = g(-2) = 0$.

Let $M = \begin{bmatrix} 1 & 1 & 3 & 1 & 1 & 1 \\ 1 & 0 & 2 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix}$, so that $M \cdot \begin{bmatrix} f(n-1) \\ g(n-1) \\ f(n-2) \\ g(n-2) \\ f(n-3) \\ g(n-3) \end{bmatrix} = \begin{bmatrix} f(n) \\ g(n) \\ f(n-1) \\ g(n-1) \\ f(n-2) \\ g(n-2) \end{bmatrix}$
for all $n \geq 1$.

Let F be the generating function

$$\sum_{N=0}^{\infty} f(N)\ x^N.$$

$f(N) = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \cdot M^N \cdot \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}^T$, which is the same as the top left entry of M^N. Thus

$F =$ the top left entry of $\sum_{N=0}^{\infty} M^N x^N = \sum_{N=0}^{\infty} (xM)^N$. Since this is a power series,

we conclude that $F =$ the top left entry of $(I - xM)^{-1} = \frac{1}{x^5 - 4x^3 - 5x^2 - x + 1} \times$

$$\begin{bmatrix} -x^3 - x^2 + 1 & (x^2 + x + 1)x & -(x^3 - 3x - 3)x & (1+x)^2 x & x(1+x) & x(1+x) \\ (1+x)^2 x & -(x^2 + 2x - 1)(1+x) & 2x(1+x) & -(1+x)^2(x-1)x & -(1+x)(x-1)x & -(1+x)(x-1)x \\ -(x^3 + x^2 - 1)x & x^2(x^2 + x + 1) & -x - 2x^2 + 1 - x^3 & (1+x)^2 x^2 & x^2(1+x) & x^2(1+x) \\ (1+x)^2 x^2 & -x(x^2 + 2x - 1)(1+x) & 2x^2(1+x) & -(x^3 + 2x^2 + 2x - 1)(1+x) & -(x-1)x^2(1+x) & -(x-1)x^2(1+x) \\ -x^2(x^3 + x^2 - 1) & x^3(x^2 + x + 1) & -x(x^3 + 2x^2 + x - 1) & (1+x)^2 x^3 & (x^4 - 3x^2 - 2x + 1)(1+x) & x^3(1+x) \\ (1+x)^2 x^3 & -x^2(x^2 + 2x - 1)(1+x) & 2x^3(1+x) & -x(x^3 + 2x^2 + 2x - 1)(1+x) & -(x-1)x^3(1+x) & -(1+x)^2(3x-1) \end{bmatrix}$$

Finally,

$$F = \frac{-x^3 - x^2 + 1}{x^5 - 4x^3 - 5x^2 - x + 1}.$$