A Reciprocity Sequence for Perfect Matchings of Linearly Growing Graphs

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The aim of this paper is to prove the following result. Let G be a finite graph. Define G_n to be the graph whose vertex set is the set of ordered pairs (v,k), $v \in V(G)$, $1 \leq k \leq n$ and where there is an edge from (v,k) to (w,l) iff either v is adjacent to w in G and $k=l$ or $v=w$ and $|k-l|=1$. We refer to the first kind of edge as a vertical edge and the second as a horizontal edge. Let P, Q be subsets of $V(G)$ and let $R(G,n,P,Q)$ be the graph formed by removing $\{(v,1), v \in P\}$ and $\{(w,n), w \in Q\}$ from G_n. This is a special case of the results in Propp's paper "A Reciprocity Theorem for Domino Tilings", http://www.math.wisc.edu/ propp/recip.ps, but do not seem to be easily derivable using the graphical methods of that paper.

Let $F(G,n,P,Q)(x)$ be the generating function in which the coeficent of x^k is the number of perfect matchings of $R(G,n,P,Q)$ with k vertical edges. For fixed G, P and Q, $F(G,n,P,Q)(x)$ will be shown to obey a linear recurrence (with coefficents in $\mathbf{Q}[x]$). This allows us to define $F(G,n,P,Q)(x)$ for n negative. Let $\overline{P}$ and $\bar{Q}$ denote the complements of P and Q. Then our central claim is

$$F(G,n,P,Q)(x) = F(G,-n,\overline{P},\overline{Q})(-x).$$

Moreover, the F are always either even or odd polynomials, so

$$F(G,n,P,Q)(x) = \pm F(G,-n,\overline{P},\overline{Q})(x)$$

where the $\pm$ is given by the following table

$$\begin{array}{rrl} n \equiv 0(\bmod 2), & p-q \equiv 0(\bmod 4): & + \\ n \equiv 0(\bmod 2), & p-q \equiv 2(\bmod 4): & - \\ n \equiv 1(\bmod 2), & g-p-q \equiv 0(\bmod 4): & + \\ n \equiv 1(\bmod 2), & g-p-q \equiv 2(\bmod 4): & - \end{array}$$

with $g = V(|G|)$, $p=|P|$, $q=|Q|$. The cases not appearing in the table are those that have an odd number of vertices and thus can not be tiled.

To make this statement perfectly true, we need to define the boundary cases correctly. $F(G,1,P,Q)(x)$ is the generating function for perfect matchings of $G \setminus (P \cup Q)$ if $P \cap Q = \emptyset$ and 0 otherwise. $F(G,0,P,Q)$ is 1 if $P=Q$ and 0 otherwise.

Claim 1: Let M be an invertible h by h matrix and fix i, j. Let x_n be the sequenece whose nth term is $(M^n)_{(ij)}$. The x_n's obey a linear recurrence. Moreover, if we use any linear recurrence obeyed by x_n to extend x_n to n negtaive, then $X_{-n} = (M^{-n})_{(ij)}$.

Proof: Let the characteristic polynomial of M be $M^h - a_{h-1}M^{h-1} - \ldots - a_0$. Then $M^{n+h} = a_{h-1}M^{n+h-1} + \ldots + a_0 M^n$ and we get $x_{n+h} = a_{h-1}x_{n+h-1} + \ldots + a_0 x_n$. Thus, the a_n obey a linear recurrence and this recurrence clearly continues to work for n negative. Now suppose the x_n obey some other recurrence, we must show that this recurrence gives the same extension to n negative. So suppose x_n, x'_n both obey some linear recurrence (not necesarily the same one) and $x_n = x'_n$ for $n > 0$. It is well known that a seqeunce obeys a linear reccurence relation iff it is a sum of terms of the form $n^k \alpha^n$. As the difference of such functions is another such function, we get that $d_n = x_n - x'_n$ also obeys some linear recurrence, say $\sum_{i=0}^{k} a_i d_{n+i} = 0$ with $a_0 \neq 0$. Then it follows by induction on n that $d_{-n} = 0$ so $x_n = x'_n$.

Let V be the vector space of dimension $2^{|V(G)|}$ where we label the basis vector by subsets of $V(G)$. We will find a map $M : V \to V$ such that $F(G,n,P,Q)(x) = (M^n)_{PQ}$ to which we can apply Claim 1. Define $S : V \to V$ to be the permutation matrix taking P to $\overline{P}$. For each edge (u,v) of G, define $D_{(u,v)}(x)$ to be the map that takes P to $P \setminus \{u,v\}$ if $u, v \in P$, 0 otherwise.

Claim 2: The $D_{(u,v)}$ commute and $D^2_{(u,v)} = 0$.

Proof: Consider two edges (u,v), (u',v') and some subset P of G. We want to understand $D_{(u,v)}D_{(u',v')}P$. If u, v, u' and v' are all distinct and $u,v,u',v' \in P$ then this is $P \setminus \{u,v,u',v'\}$. If not, the result is 0. As this description is symmetric in interchanging (u,v) with (u',v'), the matrices commute. If $(u,v)=(u',v')$ then clearly the vertices are not distinct, so we always get 0.

Claim 3:
$$F(G,n,P,Q)(x)=\left(\left(\prod_{(u,v)\in G}(1+xD_{(u,v)})\right)S\right)^n_{PQ}.$$

Proof: We will abreviate $A=\prod_{(u,v)\in G}(1+xD_{(u,v)})$ and $M=AS$. Our proof is by induction on n. The result is clear for $n=0$. For those to whom it is not clear that the $n=0$ definition is compatible with induction, we also check $n=1$. If $P\cap Q\neq\emptyset$, then there is a point in P not in $\overline{Q}$. But $S(Q)=\overline{Q}$ and every graph occuring in $A\overline{Q}$ is obtained from $\overline{Q}$ by deleting vertices, so P does not occur in $M(Q)$. If, on the other hand, $P\cap Q=\emptyset$, then $F(G,1,P,Q)(x)$ counts the tilings of $G\setminus(P\cup Q)=\overline{Q}\setminus P$. This is the same as the number of ways to remove edges from $\overline{Q}$ and leave P, which is exactly what the coefficent of P in $A\overline{Q}$ is.

Now for the inductive step. For any perfect matching of $R(G,n,P,Q)$, let $P'\subseteq\overline{P}$ be the set of $v\in G$ such that $(v,1)$ is covered by a horizontal edge. The generating fnuctin for matchings for a partiucalr P' is the number of ways to obtain P' from $\overline{P}$ by deleting edges of G, times x raised to the number of deleted edges, times $F(G,n-1,P',Q)$. (The P' occurs because the points $(v,2)$, $v\in P'$ are already covered by the horizontal edges from $(v,1)$ so we are left with a $R(G,n-1,P',Q)$ to match.) So
$$F(G,n,P,Q)(x)=\sum_{P'}\left(\begin{array}{c}\text{number of ways to delete}\\ \text{disjoint edges from } \overline{P} \text{ and get } P'\end{array}\right)x^{\text{number of edges removed}}F(G,n-1,P',Q)(x).$$

This is exactly the recurrence from taking powers of M.

So, combining Claims 1 and 2, we know that
$$F(G,-n,\overline{P},\overline{Q})(-x)=(M(-x)^{-n})_{\overline{PQ}}=(SM(-x)^{-n}S^{-1})_{PQ}=(SM(-x)^{-1}S^{-1})^n_{PQ}.$$
We want to show this is the same as $(M(x)^n)_{PQ}$. It obviously suffices to show
$$M(x)=SM(-x)^{-1}S^{-1}$$
This is the same as
$$A(x)S=S(A(-x)S)^{-1}S^{-1}=A(-x)^{-1}S^{-1}$$
Noting the obvious fact that $S=S^{-1}$, we are reduced to showing that $A(x)^{-1}=A(-x)$. As the $D_{(u,v)}$ copmmute, it suffices to show $(1+xD_{(u,v)})^{-1}=1-xD_{(u,v)}$. This is easy, $(1+xD_{(u,v)})(1-xD_{(u,v)})=1-x^2D^2_{(u,v)}=1$ as we showed $D^2_{(u,v)}=0$.

We have now proved the first form of the recirocity relationship. We still need to show the $F(G,n,P,Q)(x)$ are always either even or odd and have the sign claimed by the table at the beginning of the paper. It is equivalent to show that the parity of the number of vertical edges in any tiling of $R(G,n,P,Q)$ is independent of the choice of matching. We will set g, p, q to be the sizes of G, P and Q respectively. The total number of edges, horizontal and vertical, used in any matching $(ng-p-q)/2$.

For every horizontal edge $((v,k),(v,k+1))$, exactly one of k and $k+1$ are even, so the number of horizontal edges is the same as the number of (v,l) with l even that occur as an endpoint of a horizontal edge. For any given l, the number of (v,l) which are endpoints of vertical edges is even, so the parity of the number of hroizontal edges with endpoints of that form is simply the number of total points of the form (v,l). This is $g-p$ if $l=1$, $g-q$ if $l=n$ and g otherwise. So the parity of the number of horizontal edges is that of $((n-1)/2)g$ if n is odd and of $ng/2-q$ if n is even. So the parity of the number of vertical edges is
$$\begin{array}{ll}(ng-p-q)/2-(n-1)g/2=(g-p-q)/2 & n\equiv 1(\bmod 2)\\ (ng-p-q)/2-(ng/2-q)=(q-p)/2 & n\equiv(0 \bmod 2)\end{array}$$

This matches the table above, and our proof is complete.