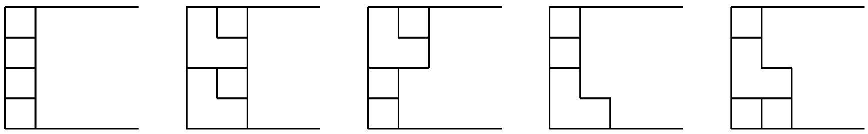
Solution to Problem 10877 of *The American Mathematical Monthly*

Bridget Eileen Tenner

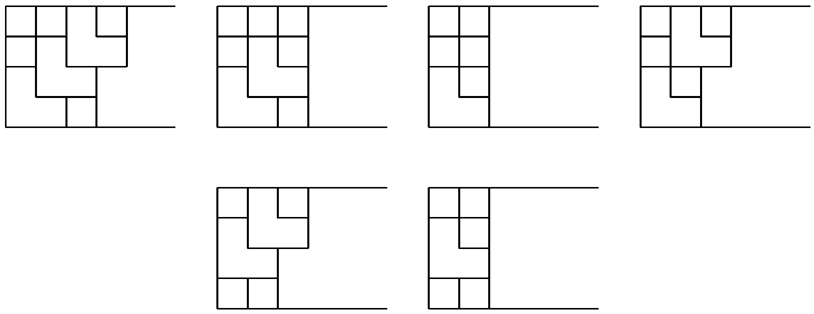
October 17, 2001

Problem. An L-tile is a 2-by-2 square with the upper right 1-by-1 subsquare removed; no rotations are allowed. Let a_n be the number of tilings of a 4-by-n rectangle using tiles that are either 1-by-1 squares or L-tiles. Find a closed form for the generating function $1 + a_1x + a_2x^2 + \ldots$.

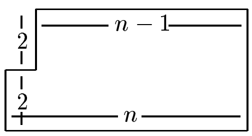
Solution. If we look at the leftmost columns of a 4-by-n rectangle, we can explicitly describe all the ways to tile these with 1-by-1 squares and L-tiles:



We can look at a few more columns to be even more precise about what must occur in the fourth and fifth examples above.



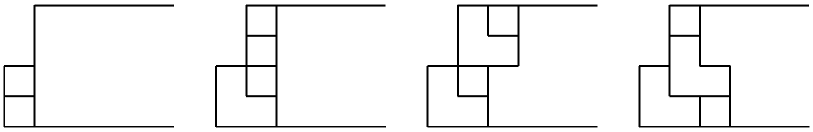
Thus any L-tiling of a 4-by-n rectangle must fall into one of these categories: the first, second, or third of the top row of diagrams, or any in the second or third rows of diagrams. If we let b_n be the number of tilings of the rectilinear region:



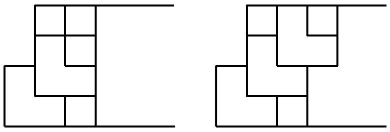
by 1-by-1 squares and L-tiles, then we have the relation:

$$a_n = a_{n-1} + 3a_{n-2} + a_{n-3} + b_{n-1} + 2b_{n-2} + b_{n-3}. \tag{1}$$

We can similarly categorize all tilings of the rectilinear figure above, by again considering how the leftmost columns must be tiled:



Notice that the fourth of these diagrams must have one of the following two forms:



Thus we also get the relation:

$$b_n = a_{n-1} + a_{n-2} + a_{n-3} + b_{n-2} + b_{n-3}. \tag{2}$$

Define $a_0 = 1$, $a_{-1} = a_{-2} = 0$, and $b_0 = b_{-1} = b_{-2} = 0$. Combinatorially these are the logical choices since not all the reductions are valid for small values of n, and these definitions allow Equations (1) and (2) to hold for all $n > 0$.

If we let $A(x) = \sum_{n=0}^{\infty} a_n x^n$ and $B(x) = \sum_{n=0}^{\infty} b_n x^n$, we can find a closed form for $A(x)$ using standard techniques for solving recurrence relations:

Multiply Equation (1) by x^n and sum both sides for $n \geq 3$. This gives:

$$\sum_{n=3}^{\infty} a_n x^n = x\sum_{n=2}^{\infty} a_n x^n + 3x^2\sum_{n=1}^{\infty} a_n x^n + x^3\sum_{n=0}^{\infty} a_n x^n + x\sum_{n=2}^{\infty} b_n x^n + 2x^2\sum_{n=1}^{\infty} b_n x^n + x^3\sum_{n=0}^{\infty} b_n x^n. \tag{3}$$

Given the definitions of $A(x)$ and $B(x)$, we can now simplify this to:

$$\begin{aligned} A(x)(1 - x - 3x^2 - x^3) &= B(x)(x + 2x^2 + x^3) + (a_0 + a_1 x + a_2 x^2) - x(a_0 + a_1 x) \\ &\quad -3x^2(a_0) - x(b_0 + b_1 x) - 2x^2(b_0). \end{aligned}$$

A similar procedure with Equation (2) yields:

$$B(x)(1 - x^2 - x^3) = A(x)(x + x^2 + x^3) + (b_0 + b_1 x + b_2 x^2) - x(a_0 + a_1 x) - x^2(a_0) - x^2(b_0).$$

Using the Equations (1) and (2), we find that $a_1 = 1$, $a_2 = 5$, $b_1 = 1$, and $b_2 = 2$, so we can simplify these further to obtain:

$$A(x)(1 - x - 3x^2 - x^3) = B(x)(x + 2x^2 + x^3) + 1; \tag{4}$$

$$B(x)(1 - x^2 - x^3) = A(x)(x + x^2 + x^3). \tag{5}$$

Solving for $B(x)$ in Equation (5) and substituting this into Equation (4) gives the desired generating function:

$$A(x) = \frac{1 - x^2 - x^3}{1 - x - 5x^2 - 4x^3 + x^5} = \frac{1 - x^2 - x^3}{(1 + x)(1 - 2x - 3x^2 - x^3 + x^4)}. \tag{6}$$