Tilings? *Again?*

WOW (Working On What) seminar
Department of Mathematics
UMass Lowell

Jim Propp
October 23, 2024

All slides available at

http://faculty.uml.edu/jpropp/wow24.pdf

Ground rule

This is meant to be an informal seminar.

Please interupt with questions!

I. The last 50 years

(my career in mathematics from high school onward)

Martin Gardner's Mathematical Recreations

3. THE MUTILATED CHESSBOARD

THE PROPS FOR this problem are a chessboard and 32 dominoes. Each domino is of such size that it exactly covers two adjacent squares on the board. The 32 dominoes therefore can cover all 64 of the chessboard squares. But now suppose we cut off two squares at diagonally opposite corners of the board [*see Fig. 13*] and discard one of the dominoes. Is it possible to place the 31 dominoes on the board so that all the remaining 62 squares are covered? If so, show how it can be done. If not, prove it impossible.

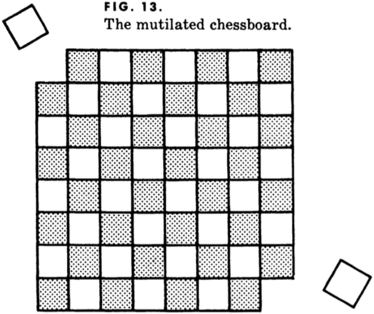
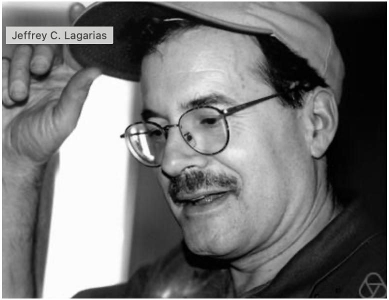


FIG. 13.
The mutilated chessboard.

Martin Gardner's Mathematical Recreations

3. It is impossible to cover the mutilated chessboard (with two opposite corner squares cut off) with 31 dominoes, and the proof is easy. The two diagonally opposite corners are of the same color. Therefore their removal leaves a board with two more squares of one color than of the other. Each domino covers two squares of opposite color, since only opposite colors are adjacent. After you have covered 60 squares with 30 dominoes, you are left with two uncovered squares of the same color. These two cannot be adjacent, therefore they cannot be covered by the last domino.

Jeff Lagarias's 1985 (1987?) talk



Conway and Lagarias's 1990 paper

Tiling with Polyominoes and Combinatorial Group Theory

J. H. Conway

Princeton University, Princeton, New Jersey

AND

J. C. Lagarias

AT&T Bell Laboratories, Murray Hill, New Jersey

Communicated by Andrew Odlyzko

Received May 3, 1988

Triangular honeycomb regions

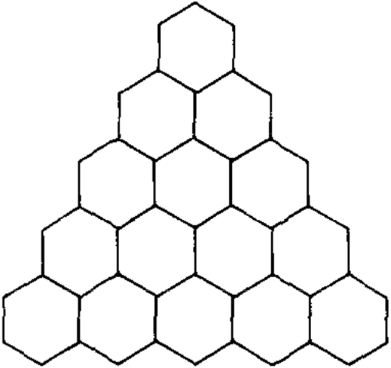


FIG. 1.2. Triangular region T_5.

Tiling honeycomb triangles

THEOREM 1.1. *The triangular region T_N in the hexagonal lattice can be tiled by congruent copies of the triangular tile T_2 if and only if*

$$N \equiv 0,\ 2,\ 9,\ \textit{or}\ 11 \pmod{12}.$$

THEOREM 1.2. *It is impossible to tile the triangular region T_N in the hexagonal lattice with congruent copies of the three-in-line tile L_3.*

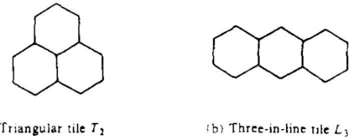


FIG. 1.3. Tiles for triangle tiling problems.

An application to domino tilings

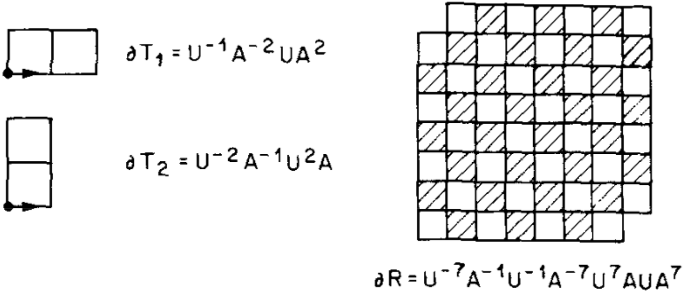


FIG. 5.1. Mutilated checkerboard and dominoes.

Thurston's 1990 paper

Conway's Tiling Groups

WILLIAM P. THURSTON, *Princeton University*

DR. THURSTON's numerous distinctions and honors include the Oswald Veblen Prize in Geometry, the Alan T. Waterman Award of the NSF, and the Fields Medal. His Ph.D. is from Berkeley, in 1972.

1. Introduction

John Conway discovered a technique using infinite, finitely presented groups that in a number of interesting cases resolves the question of whether a region in the plane can be tessellated by given tiles. The idea is that the tiles can be interpreted as describing relators in a group, in such a way that the plane region can be tiled, only if the group element which describes the boundary of the region is the trivial element 1.

Thurston's 1990 paper

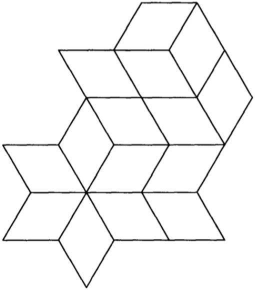


FIG. 2.1. A region tiled by lozenges. A portion of an equilateral triangle subdivision of the plane, tiled by lozenges.

Lozenge tilings

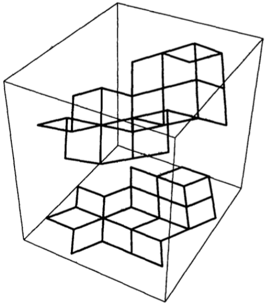
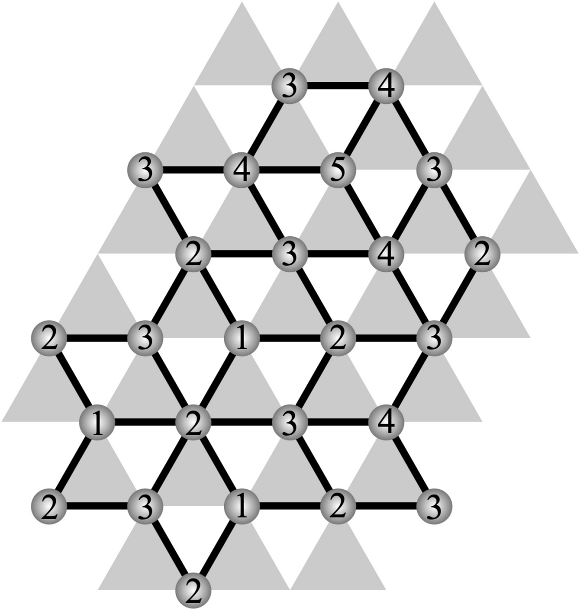


FIG. 2.2. Three-dimensional interpretation of lozenge tiling. If a region R can by tiled by lozenges, then the lozenge pattern lifts to the 2-skeleton of a cubical tiling of R^3, oriented diagonally to the plane of the lozenges.

Tilings determine height functions and vice versa



Tilings determine height functions and vice versa

From tilings to height functions:

When you travel along a tile-edge, the height goes up by one if there's a shaded triangle on your left and the height goes down by one if there's a shaded triangle on your right.

There's a unique way to do this (up to a constant).

From height functions to tilings:

Where the height changes by ± 1, draw a tile-edge.

Thurston's 1990 paper does dominoes too

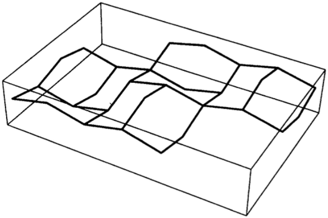
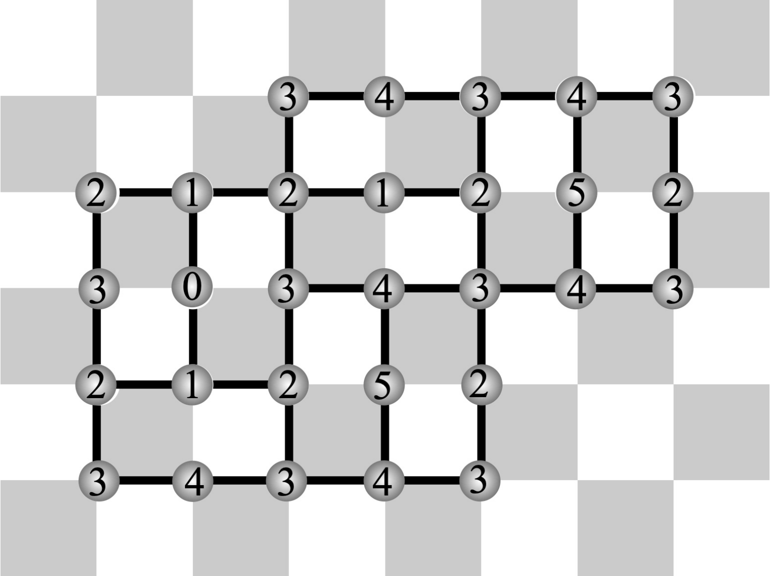


FIG. 4.3. Domino tiling. A tiling by 9 dominoes, lifted to the graph of the domino group.

Tilings determine height functions and vice versa



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From height functions to tilings:

Where the height changes by ± 1, draw a tile-edge.

The tileability problem

A priori, every finite tileability problem ("Can this region be tiled or not?") can be solved in time exponential in the area of the region being tiled.

Thurston used height functions to provide a linear time algorithm for lozenges and dominoes.

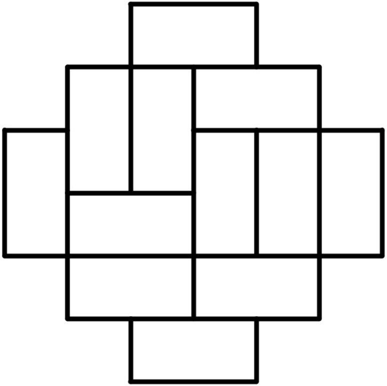
I'm a combinatorialist

MacMahon's 1916 theorem (rephrased)

The number of lozenge tilings of a regular hexagon of side length n is

$$\prod_{i=1}^{n}\prod_{j=1}^{n}\prod_{k=1}^{n}\frac{i+j+k-1}{i+j+k-2}$$

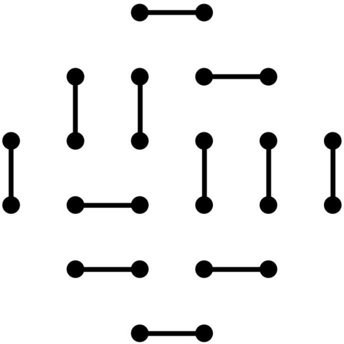
Elkies-Kuperberg-Larsen-Propp's 1988 theorem



The number of domino tilings of an "Aztec diamond" with $2n$ rows is $2^{n(n+1)/2}$.

Grensing, Carlsen, and Zapp (1980)

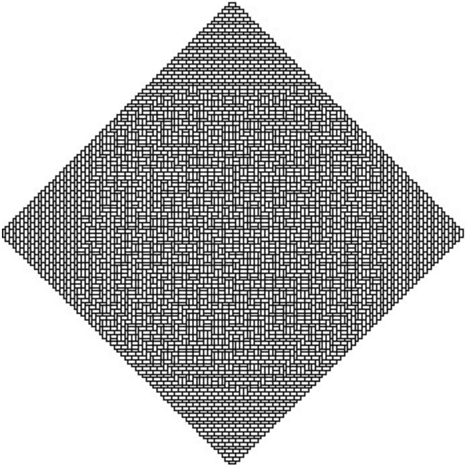
This formula was observed by physicists in the context of the dimer model on a square grid, but no proof was given.



"There's often mathematical gold in the trash cans of physicists." (Dyson? Mandelbrot?)

Into the 1990s

I got interested in random generation of these objects.



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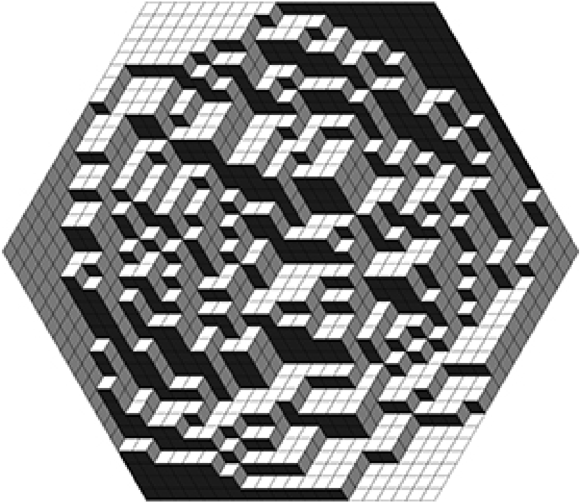


Markov Chain Monte Carlo

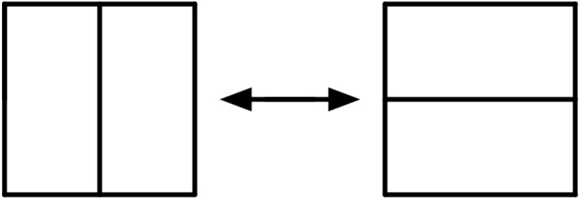
One way to generate random tilings (e.g., lozenge tilings of hexagons) is to start with non-random tilings and randomize them "for long enough".

(With Propp and Wilson's "Coupling From The Past" variant of MCMC, you can achieve randomization "for infinite time" with a finite amount of computation, but that's another story.)

But what does it mean to "randomize"?

How to randomize a domino tiling

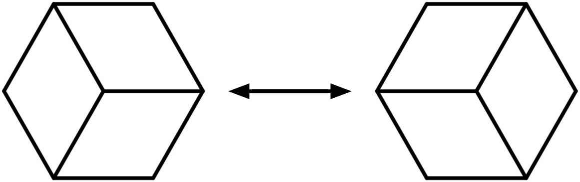
Through repeated application of “tatami moves”, any domino tiling of a simply-connected finite plane region can be converted into any other.



If one performs tatami moves randomly, the initial tiling converges in distribution to a uniform random tiling.

How to randomize a lozenge tiling

Through repeated application of "cube moves", any lozenge tiling of a simply-connected finite plane region can be converted into any other.



If one performs cube moves randomly, the initial tiling converges in distribution to a uniform random tiling.

Random tilings

With Cohn, Elkies, Jockusch, Larsen, and Shor I proved some rigorous results about the macroscopic behavior of random tilings.

(For domino tilings of Aztec diamonds and lozenge tilings of hexagons, as the size of the region grows to infinity, the boundary between the frozen region near the boundary and the jumbled region in the interior converges in law to a perfect circle!)

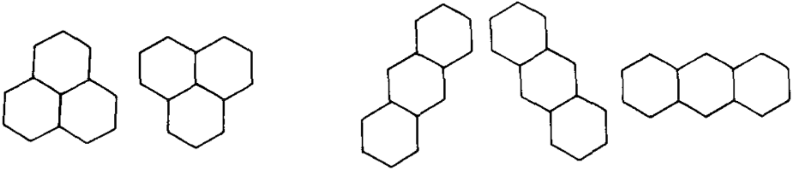
Then I got interested in other things (chip-firing, quasirandom processes, dynamical algebraic combinatorics).

II. The last 5 years

(moving on from dynamical algebraic combinatorics)

Back to Conway and Lagarias

In the 2020s I decided to take a fresh look at tilings; in particular, the work of Conway and Lagarias that had inspired me in the first place.



(a) Triangular tiles

(b) Line tiles

FIG. 1.4. Tile sets of translation-inequivalent tiles.

I renamed the T_2 tiles "stones" and the L_3 tiles "bones".

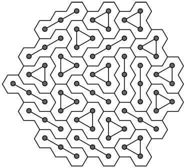
Trimers

In the language of statistical mechanics, these tiles are trimers. Physicists had given various (non-rigorously proved) asymptotic enumerative results, but no exact enumerations analogous to MacMahon's formula for lozenge tilings of hexagons or the Elkies-et-al. formula for domino tilings of Aztec diamonds.

What region should we try to tile with stones and bones, if our goal is to find exact formulas?

Benzels

I found shapes that in many ways seemed to be analogous to Aztec diamonds, and conjectured some exact formulas for the number of stones-and-bones tilings of these regions.



Working with Colin Defant, Leigh Foster, Rupert Li, and Benjamin Young I proved several of these; others were reformulated as dimer problems that in turn were solved by Seok Hyun Byun, Mihai Ciucu, and Yi-Lin Lee. See the official web page on benzels.

III. The last 0.5 years $\approx$ 200 days

(collaborating over Zoom)

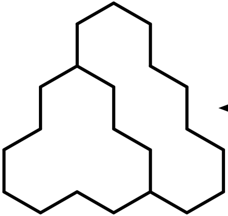
Moves for stones-and-bones tilings

Among my conjectures was a guess that I (or someone else) had made in the 1990s, regarding moves that should allow one to convert any stones-and-bones tiling t_1 of a finite simply-connected region into any other stones-and-bones tiling t_2 of that region.

Problem 19: Can every tiling of a finite simply-connected region using stones and bones be mutated into every other such tiling by means of a succession of 2-flips?

2-flips

Type I:



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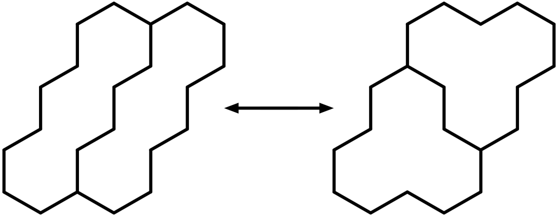


The strategy for proving the conjecture

I've refined this conjecture by proposing an empirically supported exact formula for the minimum number of moves required to turn t_1 into t_2, analogous to similar formulas that have been proved to work for domino tilings and lozenge tilings.

The strengthened conjecture

Conjecture: The moves-distance between t_1 and t_2 equals

$$\frac{1}{36}\sum_{p} ||\theta_1(p) - \theta_2(p)||_1$$

where θ_1 and θ_2 are the height functions associated with t_1 and t_2 and p ranges over the set of vertices of the tiling.

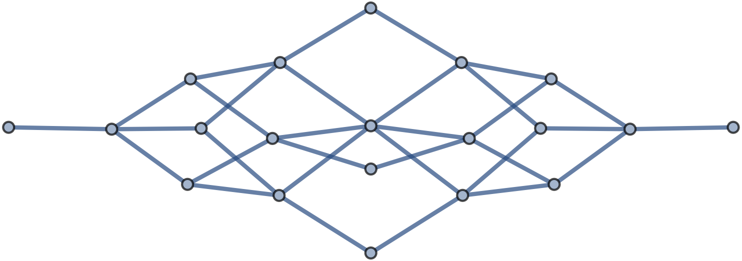
(It's not hard to show this quantity is a lower bound on the moves-distance.)

These height functions take their values not in $\mathbb{Z}$ but in the two-dimensional discrete space $\{(i,j,k) \in \mathbb{Z}^3 \mid i+j+k=0\}$.

I'll describe the relevant height-functions in a minute.

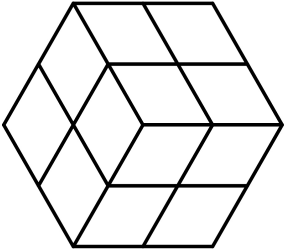
The space of lozenge tilings of a hexagon

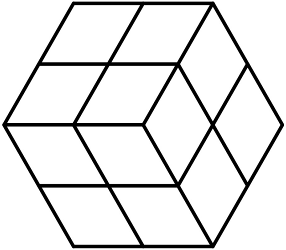
Vertices correspond to tilings, edges correspond to cube moves.



The space of lozenge tilings of a hexagon

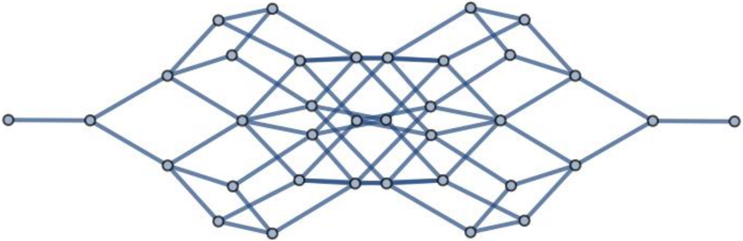
The two extremal tilings are





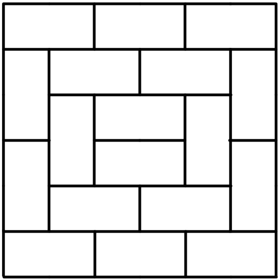
The space of domino tilings of a square

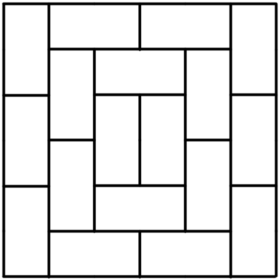
Vertices correspond to tilings, edges correspond to tatami moves.



The space of domino tilings of a square

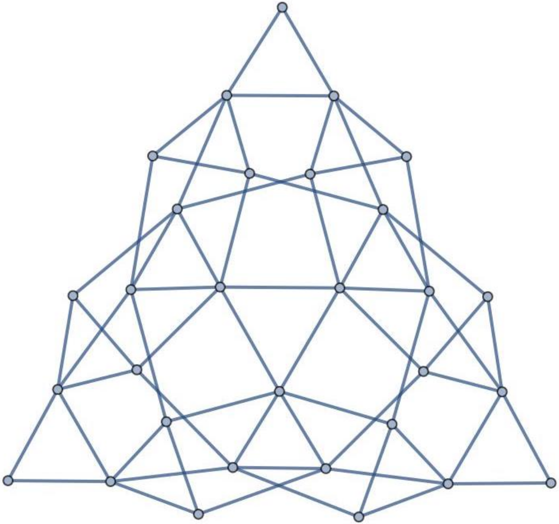
The two extremal tilings are





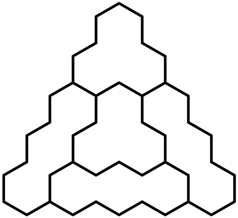
The space of stones-and-bones tilings of a triangle

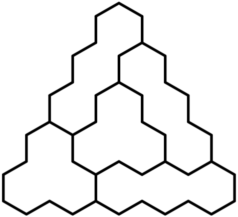
Vertices correspond to tilings, edges correspond to 2-flips.

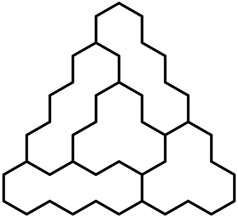


The space of stones-and-bones tilings of a triangle

The three extremal tilings are

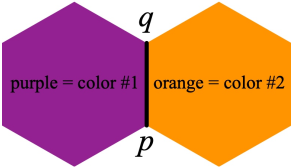






Height functions

"Height" is now a <u>triple</u> of integers that goes up by $e_i - e_j$ when you travel along a tile-edge with a color i to your left and color j to your right (with $e_1 = (1, 0, 0)$ etc.).

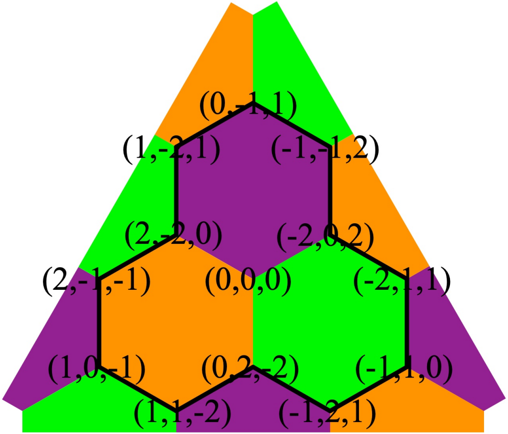


$$\theta(q) = \theta(p) + (1, -1, 0)$$

But what about vertices that don't lie on any edge?

A kludge that works

We assign each vertex in the interior of a stone a height equal to the average of the heights of its three neighbors.



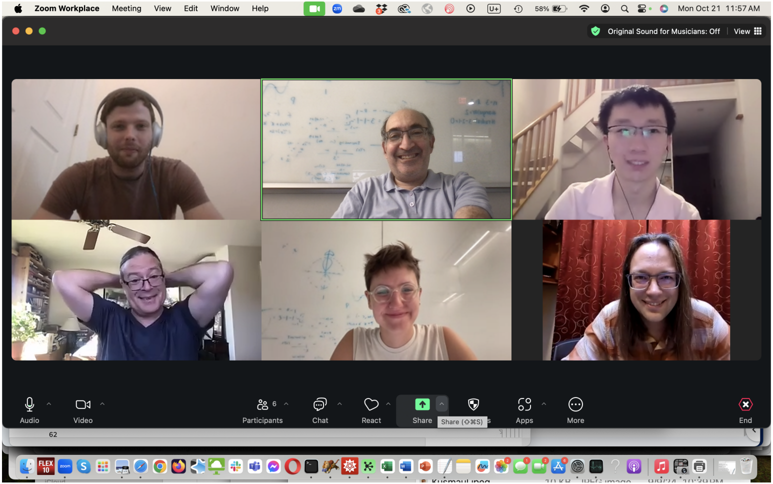
IV. The last 20 days

We've found shortest paths between the extremal tilings that validate our predictions about how far apart they are.

V. The last 2 days

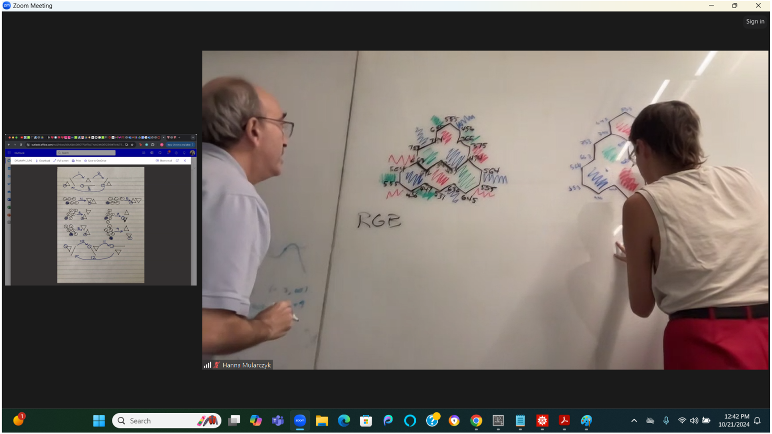
We met over Zoom to make a plan for classifying the 72 different moves you get when you take color into account (which of the 72 moves move you closer to, or further from, this-or-that extremal tiling of the triangle?).

Our intercontinental Monday meeting



Pictured: Colin Defant, Jim Propp, Rupert Li, Cris Moore, Hanna Mularczyk, and Ben Young.

An example of our process



Unable to attend, but actively involved

Our other group member, Leigh Foster, couldn't make the Zoom meeting:

VI. The future

I feel we're very close to understanding what's going on with stones-and-bones tilings of triangles and simply-connected regions in general.

I also hope we'll find an efficient algorithm for telling when a region can be tiled by stones and bones.

And: What do random stones-and-bones tilings look like? Stay tuned!

All slides available at

http://faculty.uml.edu/jpropp/wow24.pdf

Thanks for listening; I'm happy to take questions you haven't already asked.