A Review of Trigonometry

Mathematical Sciences
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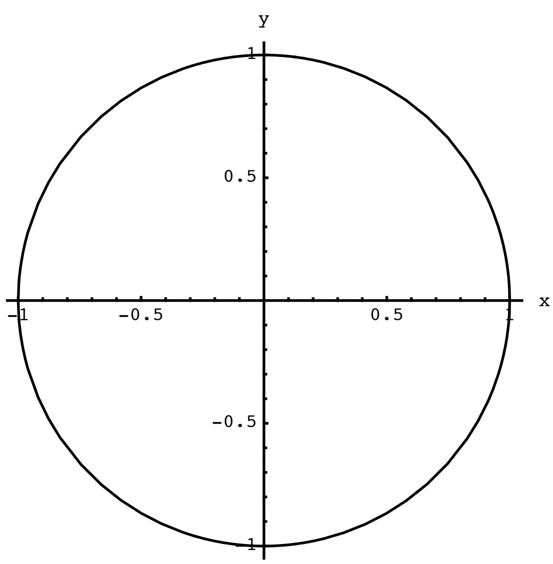
Introduction

Trigonometry is often introduced as a system of ratios of sides of right triangles. Although this aspect of the subject is useful, it is to restrictive for further uses. This document outlines the development of trigonometry as a system of "circular functions."

The unit circle

In the Cartesian coordinate system, where each point is identified as pair (x, y), the *unit circle* is the set of all points that are one unit away from the origin, (0, 0).



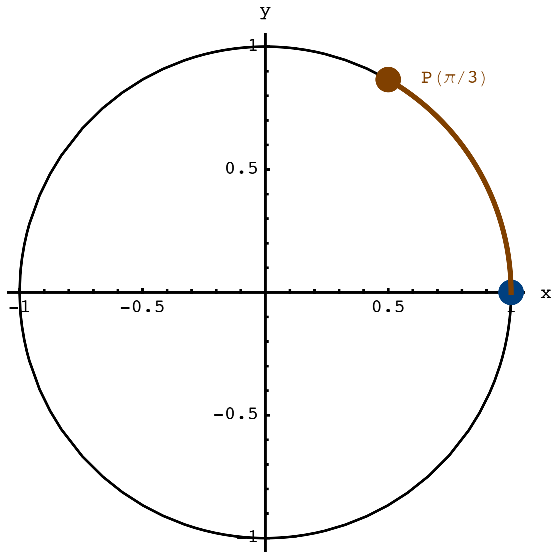
Questions

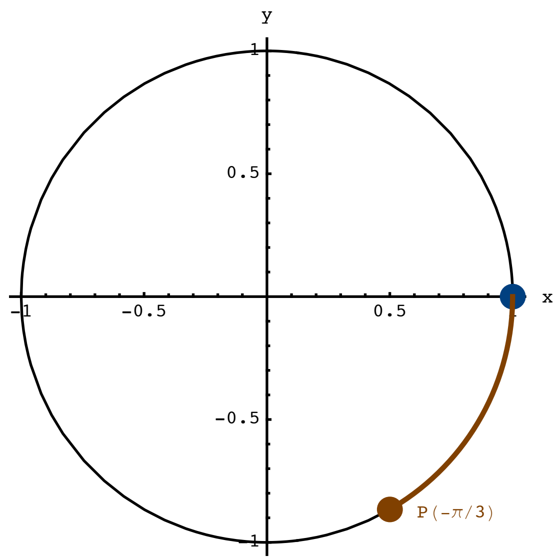
(a) What is the distance between any point $P(x, y)$ and the origin?
(b) What algebraic equality must be true in order for P to be on the unit circle?
(c) What is the circumference of the unit circle?

The wrapping function

The *wrapping function* is based on the idea that if you start with a string of length t, place one end at the point (1, 0) and wrap the string counterclockwise around the circle, the other end will lie on a point called $P(t)$. The function is defined for negative numbers by agreeing that $P(-t)$ is the terminal point of the string of length t that is wrapped clockwise starting at (1, 0). The definition is complete by stipulating that $P(0) = (1, 0)$.

Since the circumference of the unit circle is 2π, a string of length $\frac{2\pi}{6} = \frac{\pi}{3}$ will terminate at a point one sixth of the way around the circle.





■ Questions

Why does it make sense to define $P(0) = (1, 0)$?

If you extend your string by an additional 2π units from t to $t + 2\pi$, you end up at the same point since the 2π just brings you around the circle one additional rotation. This is summarized by the general identity $P(t + 2\pi) = P(t)$.

There are many other similar identities try to fill in the blanks:

(a) $P(t + 4\pi) =$ ___

(b) $P(t + \pi) =$ ___

(c) $P\left(t + \frac{\pi}{2}\right) =$ ___

Use your imagination - you might be able to come up with some other identities.

Compare the x coordinates of $P(t)$ and $P(-t)$, how are they related?

Compare the y coordinates of $P(t)$ and $P(-t)$, how are they related?

(a) If $P(t)$ lies in the second quadrant and its x coordinate is $-4/5$, what is its y coordinate?
(b) If $P(t)$ lies in the third quadrant and its y coordinate is $-5/13$, what is its x coordinate?

Notes: The units of t in the wrapping function are *radians*; so called because one unit of length is the length of the radius of the unit circle. One can think of the variable t as time, in which case $P(t)$ is capturing the motion a point that is rotating around a circle with constant speed.

The circular functions

The circular function *sin* (short for *sine*) and *cos* (short for *cosine*) are defined in terms of the wrapping function.

$\sin t$ is the y coordinate of $P(t)$
and
$\cos t$ is the x coordinate of $P(t)$

- **Questions**

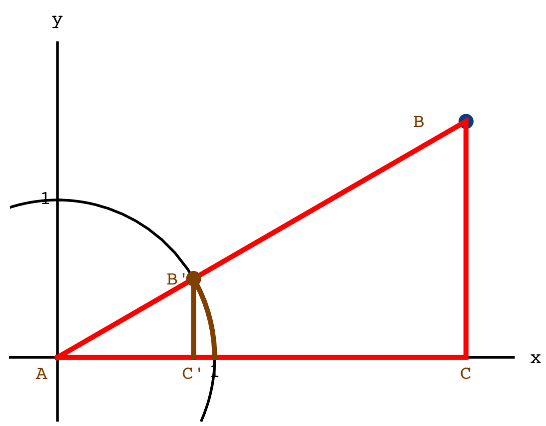
As t increases from 0 to 2π, what values does $\sin t$ takes on? What values does $\cos t$ take on?

The identity $P(t+2\pi) = P(t)$ tells us that $\sin(t+2\pi) = \sin(t)$. What other properties of the circular functions are derived directly from properties of the wrapping function?

(A calculus question - see if you can answer it intuitively) As t increases past zero, $\sin t$ increases. How fast is $\sin t$ increasing when $t = 0$?

Connection with right triangle trigonometry

Place a right triangle *ABC* in the cartesian coordinate system as shown in the figure below, with the hypotenuse emanating from the origin and lying in the first quadrant. In addition, one side lies on the positive x-axis. Let B' be the point of intersection of the hypotenuse with the unit circle and drop B' down to C' on the x-axis to form triangle $A B' C'$, which is similar to ABC. In right triangle trigonometry, the sine of $\angle \mathrm{BAC}$ is the ratio $\frac{|\mathrm{BC}|}{|\mathrm{AB}|}$. Since $A B' C'$ is similar, $\frac{|B\,C|}{|A\,B|} = \frac{|B'\,C'|}{|A\,B'|} = |B'\,C'|$. Notice that $|B'\,C'|$ is the y-axis of the point B', which is $P(t)$, where t is the length of the arc of the unit circle inside ABC - *exactly how we define* $\sin t$ as a circular function.



Thus, the only difference between the circular functions and the right triangle trigonometric functions is the angle measurement system that is used. Instead of the familiar degree system, we use radians to measure angle in which the angle $\angle \mathrm{BAC}$ is the arc length, t.

- **Question**

Use geometry to derive the values $\cos \frac{\pi}{3}$ and $\sin \frac{\pi}{4}$. Click here for the correct values.

The other circular functions

The other circular functions are defined in terms of sine and cosine. We won't do much with them here.

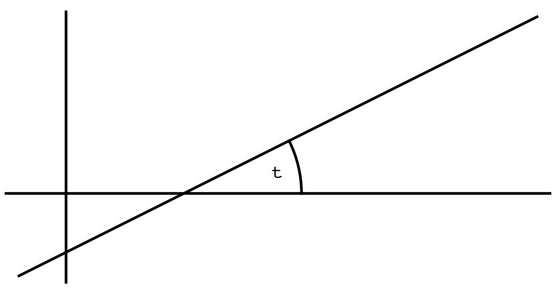
tangent: $\tan t = \frac{\sin t}{\cos t}$

cotangent: $\cot t = \frac{\cos t}{\sin t} = \frac{1}{\tan t}$

secant: $\sec t = \frac{1}{\cos t}$

cosecant: $\csc t = \frac{1}{\sin t}$

One particulary important thing about the tangent function: How does the angle a line makes with the x axis determine its slope?

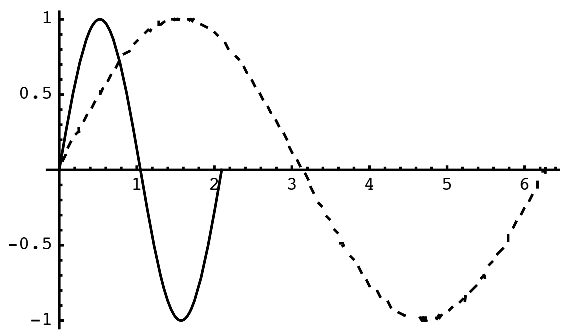


Transformations

Given any functional relationship $y = f(t)$, the effect on that relationship by replacing $f(t)$ with $f(c\,t)$, $f(t+c)$, $c\,f(t)$, $f(t)+c$, or a combination of these is away of interest. The first three are of particular interest for the circular functions and have particular names associated with them.

- **Changing the period and frequency.**

The contrast between $\sin t$ and $\sin c\,t$ is that, depending on the specific value of c, the *period* will change from 2π to a different value. For example, $\sin 3\,t$ is the y -coordinate of $P(3\,t)$. Since P will wrap around the unit circle one when $3\,t = 2\,\pi$, $t = \frac{2\pi}{3}$ is the period of $\sin 3\,t$. This can be seen from the graph of the first periods of $\sin t$ (dashed) and $\sin 3\,t$ (solid).



A smaller period implies a higher *frequency* , which measures the number of times the wrapping function point rotates around the unit circle per unit time. While $\sin t$ has frequency $\frac{1}{2\pi}$, $\sin 3t$ has a higher frequency, $\frac{3}{2\pi}$. In general, frequency $= \frac{1}{\text{period}}$.

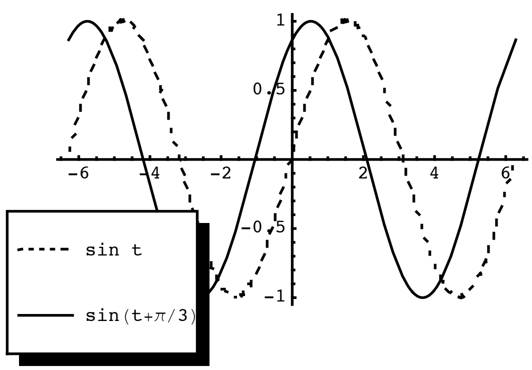
Everything said above about sine translates to cosine.

What happens when c is a relatively small number, like $\frac{1}{5}$?

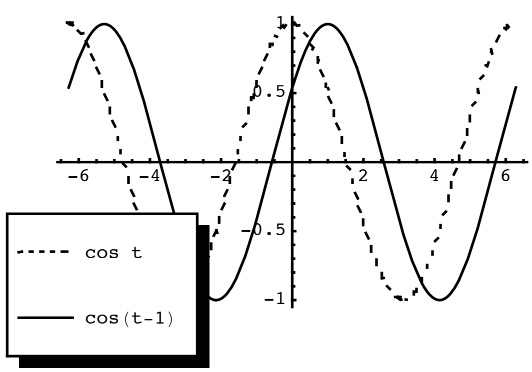
What happens when c is a negative number?

■ Phase Shifts

The contrast between $\sin t$ and $\sin\ (t + c)$ is that the latter is a shift of the former by c units to the left if $c > 0$; while the shift will be to the left in $c < 0$.

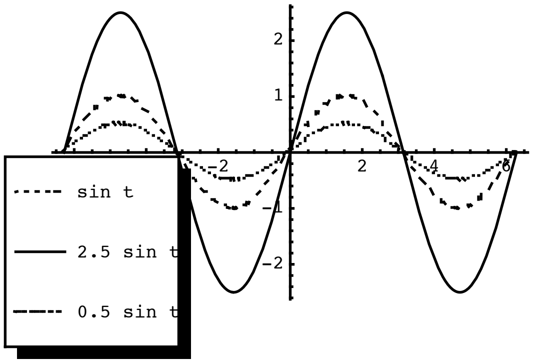


The same general rules apply to phase shifts of cosine.

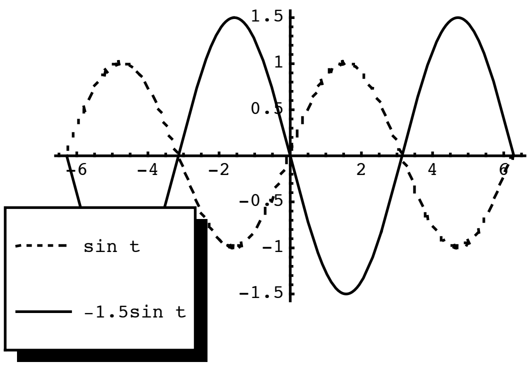


■ Change of Amplitude

Multiplying $\sin t$ by a constant will stretch or shrink the graph, depending on the constant.



If the constant is negative, the function's graph will be reflected across the *x* axis.

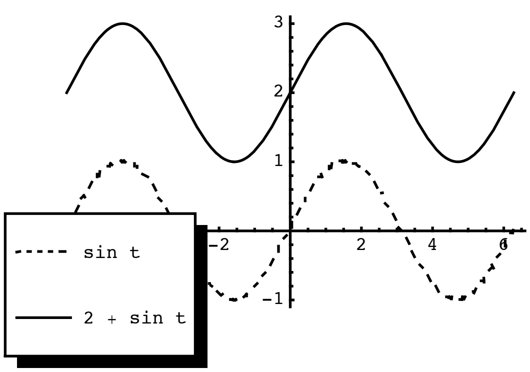


The maximum value of $|\sin t|$ is 1 — think about the unit circle! This is the *amplitude* of $\sin t$. The amplitude of $c \sin t$ is $|c|$. For example, the amplitude of $-1.5 \sin t$ is 1.5.

The situation with multiplication of $\cos t$ by a constant is exactly the same as with $\sin t$.

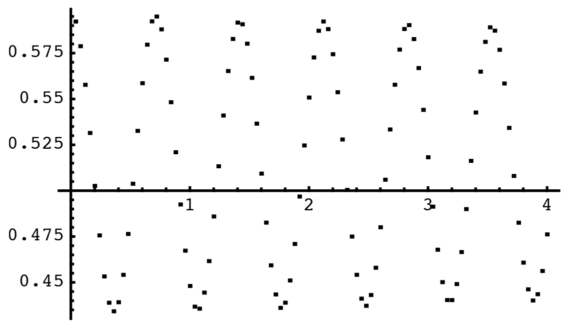
■ Vertical shifts

The $c + \sin t$ is a vertical shift of $\sin t$. Instead of ranging between -1 and 1, $c + \sin t$ will range between $c - 1$ and $c + 1$. For example, if $c = 2$:



■ Combinations

In most cases where circular functions describe real-life situations, a combination of these transformations would be needed to match a function to reality. For example, here is a plot of real data from the motion of a weight attached to a spring. The data was collected for 4 seconds and the distance of the weight from the measuring device is in meters. The data certainly *looks* like a circular function! Let's see if we can identify a circular function that fits it. We will do this be transforming the data backwards to $\sin x$.



Here is the actual data.

```
springdata = {{0.04`, 0.59222`}, {0.08`, 0.578781`}, {0.12`, 0.557728`}, {0.16`, 0.531471`},
  {0.2`, 0.502613`}, {0.24`, 0.475616`}, {0.28`, 0.453253`}, {0.32`, 0.438918`},
  {0.36`, 0.434232`}, {0.4`, 0.439177`}, {0.44`, 0.454114`}, {0.48`, 0.476408`},
  {0.52`, 0.503785`}, {0.56`, 0.532608`}, {0.6`, 0.558589`}, {0.64`, 0.579608`},
  {0.68`, 0.592323`}, {0.72`, 0.594959`}, {0.76`, 0.587964`}, {0.8`, 0.571442`},
  {0.84`, 0.548235`}, {0.88`, 0.520944`}, {0.92`, 0.492483`}, {0.96`, 0.467294`},
  {1, 0.44805`}, {1.04`, 0.436765`}, {1.08`, 0.435679`}, {1.12`, 0.444449`},
  {1.16`, 0.461609`}, {1.2`, 0.485936`}, {1.24`, 0.513312`}, {1.28`, 0.540999`},
  {1.32`, 0.565239`}, {1.36`, 0.582727`}, {1.4`, 0.591651`}, {1.44`, 0.590738`},
  {1.48`, 0.58016`}, {1.52`, 0.561518`}, {1.56`, 0.536554`}, {1.6`, 0.509315`},
  {1.64`, 0.482593`}, {1.68`, 0.4593`}, {1.72`, 0.443432`}, {1.76`, 0.43611`},
  {1.8`, 0.438936`}, {1.84`, 0.451047`}, {1.88`, 0.470964`}, {1.92`, 0.496807`},
  {1.96`, 0.524666`}, {2, 0.550785`}, {2.04`, 0.572648`}, {2.08`, 0.587189`},
  {2.12`, 0.592254`}, {2.16`, 0.588085`}, {2.2`, 0.574405`}, {2.24`, 0.553679`},
  {2.28`, 0.527922`}, {2.32`, 0.500425`}, {2.36`, 0.474995`}, {2.4`, 0.454131`},
  {2.44`, 0.441141`}, {2.48`, 0.437247`}, {2.52`, 0.443122`}, {2.56`, 0.457991`},
  {2.6`, 0.480061`}, {2.64`, 0.50599`}, {2.68`, 0.533384`}, {2.72`, 0.557762`},
  {2.76`, 0.576921`}, {2.8`, 0.588154`}, {2.84`, 0.59029`}, {2.88`, 0.582658`},
  {2.92`, 0.566859`}, {2.96`, 0.544065`}, {3, 0.518222`}, {3.04`, 0.491294`},
  {3.08`, 0.467811`}, {3.12`, 0.450134`}, {3.16`, 0.440366`}, {3.2`, 0.440348`},
  {3.24`, 0.449083`}, {3.28`, 0.466536`}, {3.32`, 0.489967`}, {3.36`, 0.516258`},
  {3.4`, 0.542618`}, {3.44`, 0.564895`}, {3.48`, 0.581142`}, {3.52`, 0.589067`},
  {3.56`, 0.587241`}, {3.6`, 0.576817`}, {3.64`, 0.558451`}, {3.68`, 0.534245`},
  {3.72`, 0.50804`}, {3.76`, 0.482542`}, {3.8`, 0.460782`}, {3.84`, 0.446206`},
  {3.88`, 0.440124`}, {3.92`, 0.443553`}, {3.96`, 0.456216`}, {4, 0.476167`}}
```

First, lets identify the maximum and minimum values:

```
{miny, maxy} = springdata // Transpose // Last // {Min[#], Max[#]} &
```

{0.434232, 0.594959}

Half-way between the two values is the value we want to transform to zero.

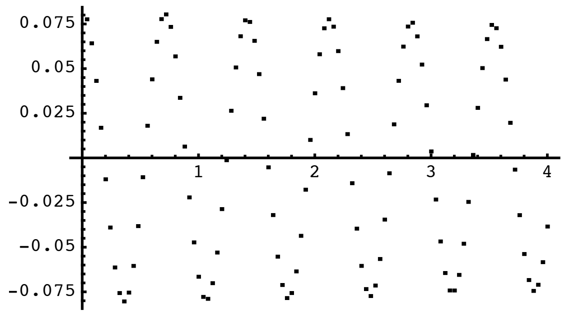
$$yt = \frac{maxy + miny}{2}$$

0.514596

If we subtract this value from each of the original *y* values, the graph will be centered at zero.

```
springdata2 = springdata /. {{a_, b_} → {a, b - yt}};
```

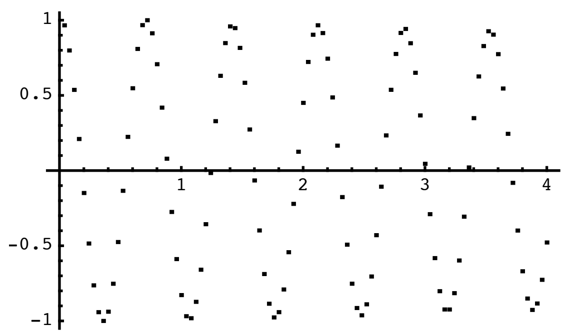
```
ListPlot[springdata2]
```




```
amplitude = maxy - yt
```

0.0803635

```
springdata3 = springdata2 /. {{a_, b_} → {a, b / amplitute}};
ListPlot[springdata3]
```



- Graphics -

```
springdata3[[1]]
```

{0.04, 0.965917}

```
springdata3[[17]]
```

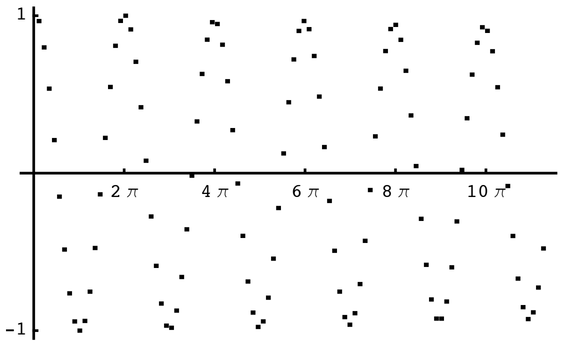
{0.68, 0.967199}

```
period = 0.71
```

0.71

```
springdata4 = springdata3 /. {{a_, b_} → {2 Pi a / period, b}};
```

```
ListPlot[springdata4, Ticks → {Range[0, 12 Pi, 2 Pi], {-1, 1}}]
```

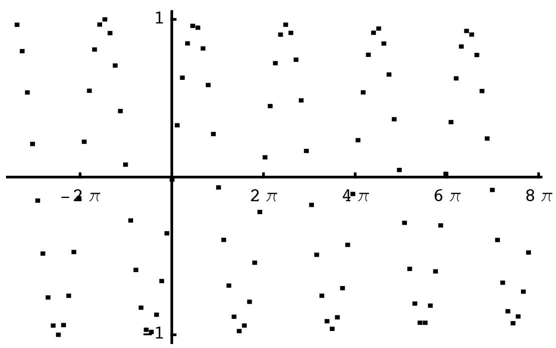


- Graphics -

```
shift = 10.9735;
```

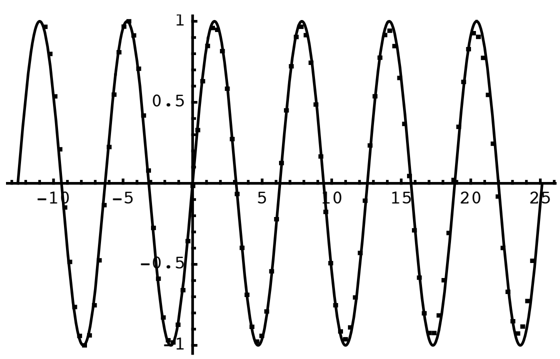
```
springdata5 = springdata4 /. {{a_, b_} → {a - shift, b}};
```

```
ListPlot[springdata5, Ticks → {Range[-2 Pi, 8 Pi, 2 Pi], {-1, 1}}]
```



- Graphics -

```
Plot[Sin[t], {t, -4 Pi, 8 Pi}, Epilog → Point /@ springdata5]
```

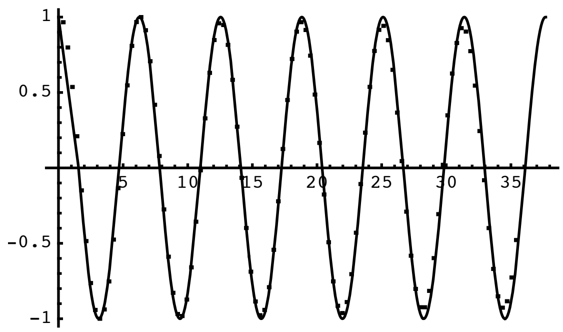


- Graphics -

Now we can proceed back to find an equation of the original data.

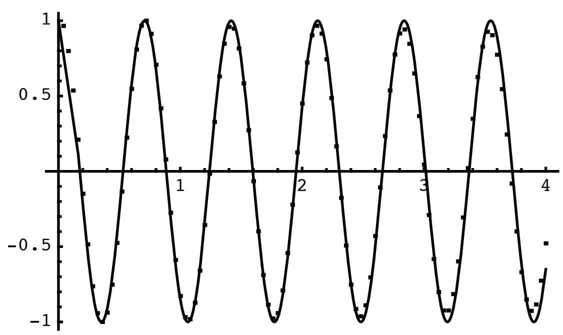
1. Adjust the phase shift.

```
Plot[Sin[t - shift], {t, 0, 12 π}, Epilog → Point /@ springdata4]
```



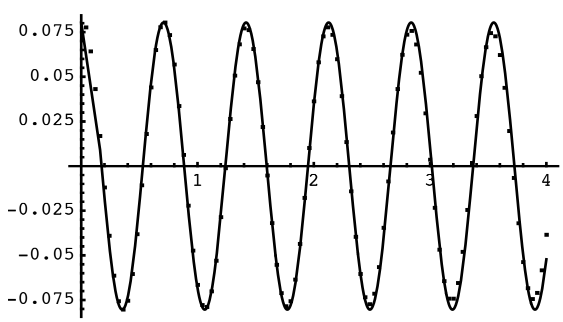
2. Adjust the period.

$$\texttt{Plot}\left[\texttt{Sin}\left[\frac{2\,\pi\,t}{\texttt{period}} - \texttt{shift}\right], \{t, 0, 4\}, \texttt{Epilog} \to \texttt{Point /@ springdata3}\right]$$



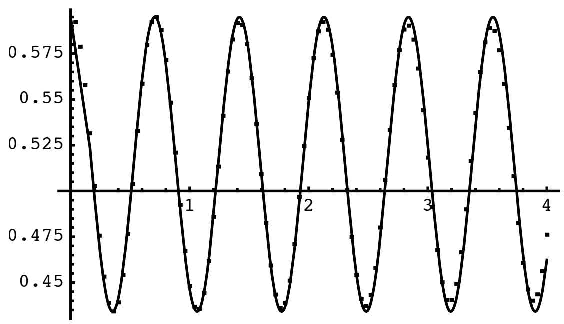
3. Adjust the amplitude.

$$\texttt{Plot}\left[\texttt{amplitute Sin}\left[\frac{2\,\pi\,t}{\texttt{period}} - \texttt{shift}\right], \{t, 0, 4\}, \texttt{Epilog} \to \texttt{Point /@ springdata2}\right]$$



3. Translate up to the mid-range.

```
Plot[amplitute Sin[(2 π t)/period - shift] + yt, {t, 0, 4}, Epilog → Point /@ springdata]
```



Therefore a reasonably close model for the original data is $y = 0.0803635\ \sin(8.84956\,t\ -\ 10.9735) + 0.514596$, as can be seen with the original data superimposed on the graph above. This formula is a good starting point for using `FindFit`, which will fine-tune the formula to take into account all the data.

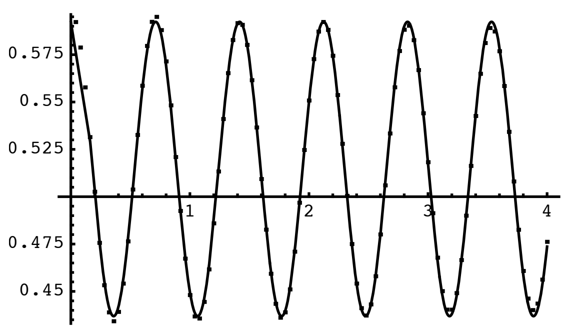
```
betterfit = a Sin[b t + c] + d /.
  FindFit[springdata, a Sin[b t + c] + d, {{a, 0.08}, {b, 8.85}, {c, -10.97}, {d, 5.15}}, t]
```

$0.51463 - 0.0777117\sin(11.0569 - 8.90854\,t)$

This result might look very different from what we had derived, but it isn't:

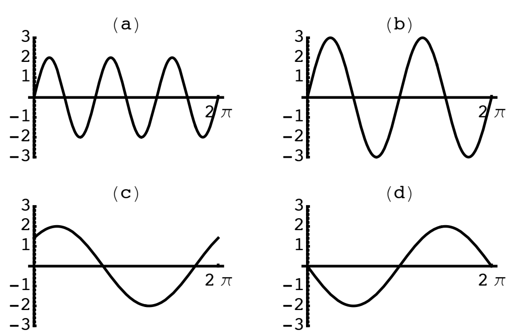
$$d\ -\ a\sin(c - b\,t)\ =\ d\ -\ a(-\sin(b\,t\ - c)) \qquad \text{since } \sin(-t) = -\sin t$$
$$= d\ +\ a\sin(b\,t - c)$$

```
Plot[betterfit, {t, 0, 4}, Epilog → Point /@ springdata]
```



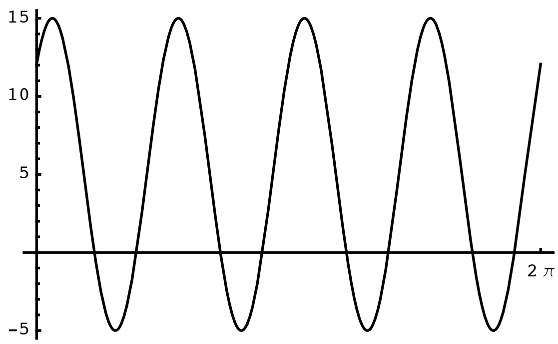
■ Questions

Match the following graphs with the circular functions listed below.



1. $2\sin\left(t + \frac{\pi}{4}\right)$
2. $3\sin(2t)$
3. $2\sin(3t)$
4. $-2\sin(t)$

Identify the amplitude, period and phase shift of the function that has the form $a\sin(b\,t + c) + d$



Identities

There are countless identities that relate various expressions involving the circular functions. The fact that very different looking expressions are really the same tends to make trigonometry confusing to many. Comments on a few examples are provided here. Several web sites provide good lists. Here are two of them:

http://en.wikipedia.org/wiki/Trigonometric_identity

http://www.sosmath.com/trig/Trig5/trig5/trig5.html

■ Basic Identities

$\sin^2 t + \cos^2 t = 1$

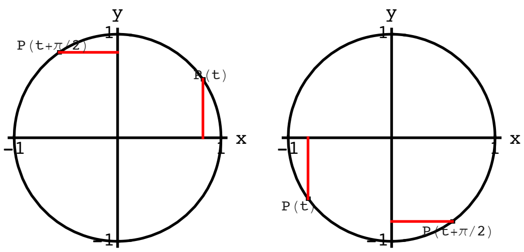
Proof: $\sin^2 t + \cos^2 t = (y \text{ coordinate of } P(t))^2 + (x \text{ coordinate of } P(t))^2$
$= 1$ since $P(t)$ is on the unit circle.

$\sin(t + \pi) = -\sin t$

Proof: $P(t)$ and $P(t + \pi)$ are at opposite sides of the unit circle and so $P(t + \pi) = -P(t)$ which means that the y coordinate of $P(t + \pi)$ is the negation of the y coordinate of $P(t)$, which is exactly what the identity states.

$\cos\left(t + \frac{\pi}{2}\right) = -\sin t$

Proof: Consider $P(t)$ and $P\left(t + \frac{\pi}{2}\right)$. If you compare the y coordinate of $P(t)$, which is $\sin t$, and the x coordinate of $P\left(t + \frac{\pi}{2}\right)$, which is $\cos\left(t + \frac{\pi}{2}\right)$, they have the same magnitude, but always opposite signs. The two images that follow are for two possible quadrants, but this works for *all* values of t.



■ More advanced identities

An example of a more advanced identity is the angle sum formula for sine:

$$\sin(t + u) = \sin t \cos u + \cos t \sin u$$

Instead of a proof (there are many that can be found in books and on the web), here is how I remember this one if I've forgotten the details.

I recall the fact that when you multiply two complex numbers, you multiply their absolute values (moduli) and add their angles (arguments). This means that if I take two complex numbers on the unit circle, $z = \cos t + \sin t\, i$ and $w = \cos u + \sin u\, i$ their product will also be on the unit circle and their product is $z\,w = (\cos t \cos u - \sin t \sin u) + (\sin t \cos u + \cos t \sin u)\, i$

At the same time, we know that $z\,w = \cos(t + u) + \sin(t + u)\, i$. Comparing the two expressions for $z\,w$, you get the identity above and the corresponding one for cosine.

Of course you need to have a facility with complex numbers to be able to recall the identities this way, but it works for me.

If you remember one of these identities, the others can be derived by using some of the more basic identities. For example,

$$\begin{aligned}\cos(u + v) &= \sin\left(u + v + \frac{\pi}{2}\right)\\ &= \sin\left(u + \left(v + \frac{\pi}{2}\right)\right)\\ &= \sin u \cos\left(v + \frac{\pi}{2}\right) + \cos u \sin\left(v + \frac{\pi}{2}\right)\\ &= \sin u\,(-\sin v) + \cos u \cos v\\ &= \cos u \cos v - \sin u \sin v\end{aligned}$$

Take some time to understand why every step is valid in deriving the identity for the cosine of the sum of two angles.

■ General Identities

There are countless more advanced and general identities. Hardly anyone has them all at their fingertips. A few are somewhat more important that others. The following has important consequences in some areas of mathematics. If n is an integer greater than 1, then

$$\cos n\,t = 2\cos(n-1)\,t\, \cos t - \cos((n-2)\,t)$$

The implication of this identity is that $\cos(n\,t)$ is a polynomial in $\cos x$. For example,

$$\begin{aligned}\cos 3\,t &= 2\cos 2\,t \cos t - \cos t\\ &= 2\left(2\cos^2 t - 1\right)\cos t - \cos t\\ &= 4\cos^3 t - 3\cos t\end{aligned}$$

Therefore, $\cos 3\,t = p(\cos t)$, where $p(x) = 4\,x^3 - 3\,x$. This particular polynomial is one called a *Chebyshev polynomial of the first kind*. These polynomials are built into *Mathematica*.

```
ChebyshevT[3, x]
```

$4\,x^3 - 3\,x$

The following demonstrates how the 7^{th} degree Chebyshev polynomial in cos t reduces to cos 7 t.

```
p = ChebyshevT[7, Cos[t]]
```

$64\cos^7(t) - 112\cos^5(t) + 56\cos^3(t) - 7\cos(t)$

```
TrigReduce[p]
```

$\cos(7\,t)$

- **Questions**

Derive the formula for the sine of the difference of two angles: $\sin(u - v)$.

Using the identities for the sine and cosine of the sum of two angles, derive identities for $\sin 2\,t$ and $\sin \frac{t}{2}$.

Find the six roots of the 6th degree Chebyshev polynomial $32\,x^6 - 48\,x^4 + 18\,x^2 - 1$

Special angles

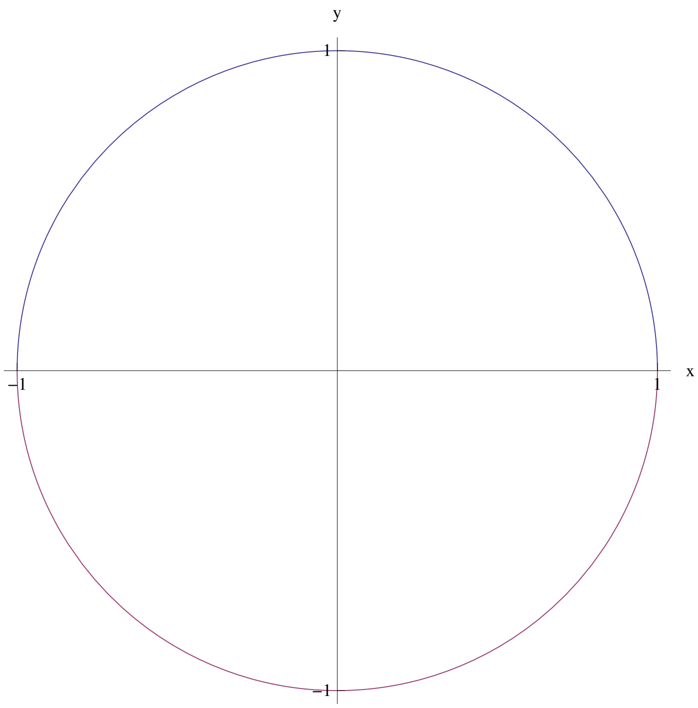
If you pick a "random" angle, it is unlikely to be one of these "special" angles, but these special cases are worth knowing by heart because they do come up in special case. For example, equilateral triangles are often the solution to certain problems.

t radians	degrees	sin t	cos t
0	0	0	1
$\frac{\pi}{6}$	30 °	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$
$\frac{\pi}{4}$	45 °	$\frac{1}{\sqrt{2}}$	$\frac{1}{\sqrt{2}}$
$\frac{\pi}{3}$	60 °	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$
$\frac{\pi}{2}$	90 °	1	0

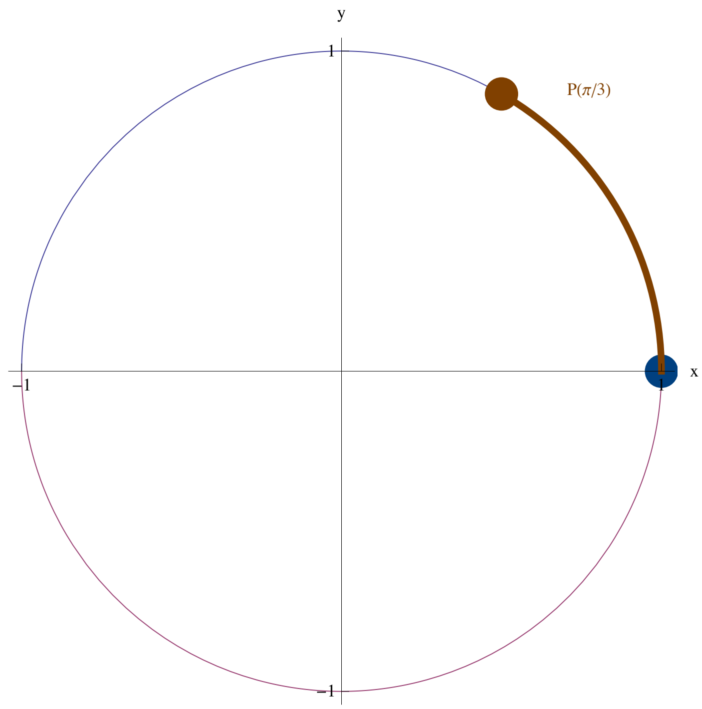
Mathematica code

This is undocumented code that was used in generating figures for this Notebook. *Mathematica* beginners should probably ignore this section. When you have more expertise with the software it might make more sense.

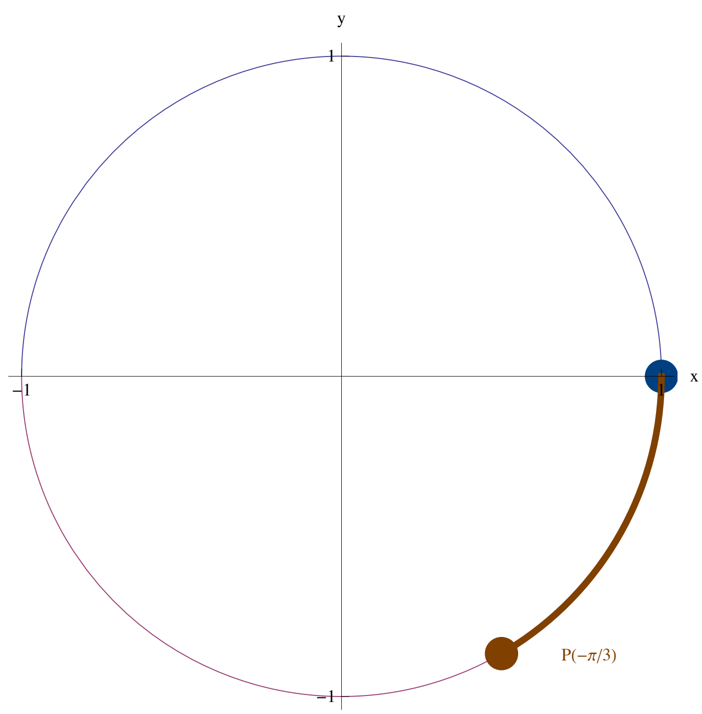
```
P[t_] := {Cos[t], Sin[t]};
unitcircle = Plot[{√(1 - x²), -√(1 - x²)}, {x, -1, 1},
  Ticks → {{-1, 1}, {-1, 1}}, AspectRatio → 1, AxesLabel → {"x", "y"}]
```



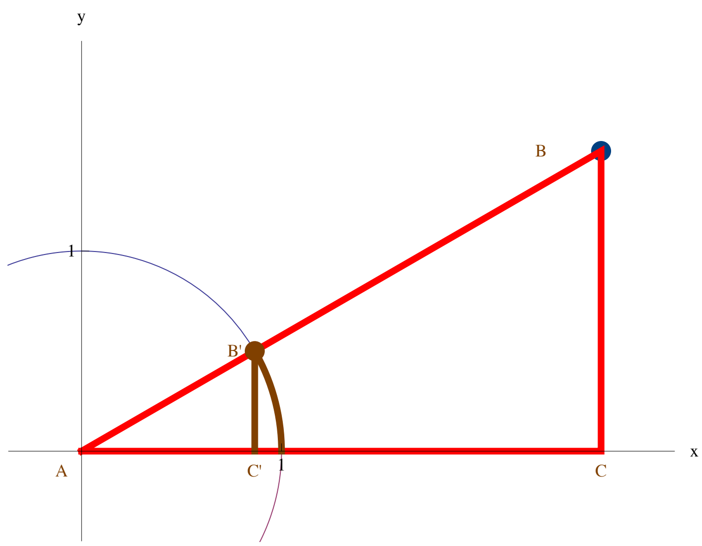
```
{unitcircle, Graphics[{RGBColor[0, 0.25098, 0.501961], PointSize[0.05], Point[P[0]],
    RGBColor[0.501961, 0.25098, 0], Thickness[0.01], Circle[{0, 0}, 1, {0, Pi / 3}],
    Point[P[Pi / 3]], Text["P(π/3)", {0.773763, 0.875517}]}]} // Show
```



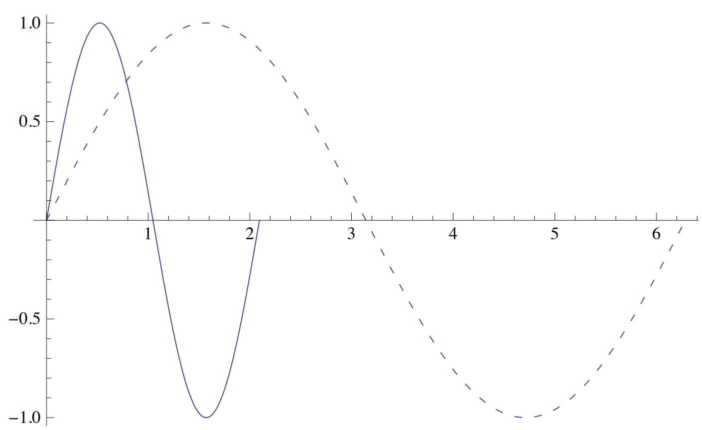

```
{unitcircle, Graphics[{RGBColor[0, 0.25098, 0.501961], PointSize[0.05], Point[P[0]],
    RGBColor[0.501961, 0.25098, 0], Thickness[0.01], Circle[{0, 0}, 1, {-Pi / 3, 0}],
    Point[P[-Pi / 3]], Text["P(-π/3)", {0.773763, -0.875517}]}]} // Show
```



```
{unitcircle, Graphics[{RGBColor[0, 0.25098, 0.501961],
   PointSize[0.03], Point[3 P[Pi / 6]], RGBColor[1, 0, 0], Thickness[0.01],
   Line[{{0, 0}, {3/2 √3, 0}, {3/2 √3, 3 / 2}, {0, 0}}], RGBColor[0.501961, 0.25098, 0],
   Line[{{√3 / 2, 0}, {√3 / 2, 1 / 2}}], Circle[{0, 0}, 1, {0, Pi / 6}], Point[P[Pi / 6]],
   Text["A", {-0.1, -0.1}], Text["C'", {√3 / 2, -0.1}], Text["C", {3/2 √3, -0.1}],
   Text["B", 3 {√3 / 2 - 0.1, 1 / 2}], Text["B'", {√3 / 2 - 0.1, 1 / 2}]}]} //
 Show[#, AspectRatio → Automatic, PlotRange → {{-0.3, 2.9}, {-0.4, 2.}}] &
```




```
g1 = Plot[Sin[t], {t, 0, 2 Pi},
   DisplayFunction → Identity, PlotStyle → Dashing[{0.01, 0.02}]];
g2 = Plot[Sin[3 t], {t, 0, 2 Pi / 3}, DisplayFunction → Identity];
Show[{g1, g2}, DisplayFunction → $DisplayFunction]
```




```
<< PlotLegends`
```