Mechanical Vibrations

Chapter 5

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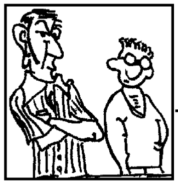


Multiple Degree of Freedom Systems

Systems with more than one DOF:

- ***Referred to as a Multiple Degree of Freedom***
- ***An NDOF system has 'N' independent degrees of freedom to describe the system***
- ***There is one natural frequency for every DOF in the system description***

Multiple Degree of Freedom Systems

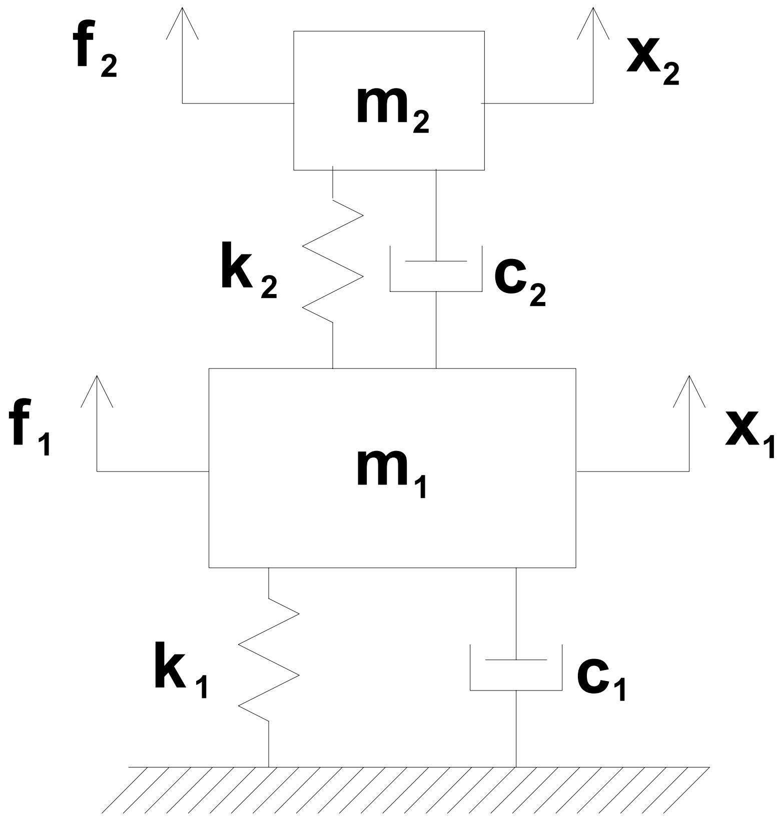
Properties of a MDOF system:

- Each natural frequency has a displacement configuration referred to as a 'normal mode'
- Mathematical quantities referred to as 'eigenvalues' and 'eigenvectors' are used to describe the system characteristics
- While the resulting motion appears more complicated, the system set of equations can always be decomposed into a set of equivalent SDOF systems for each mode of the system.

Multiple Degree of Freedom Systems

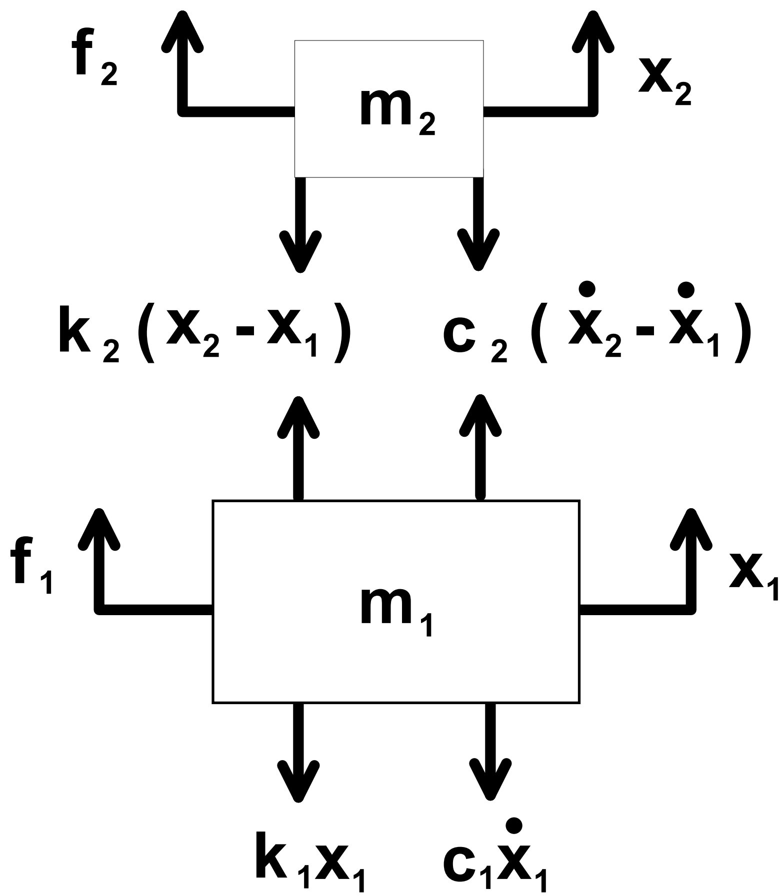
Assumptions

- lumped mass
- stiffness proportional to displacement
- damping proportional to velocity
- linear time invariant
- 2nd order differential equations



Multiple Degree of Freedom Systems

Free body diagram



Multiple Degree of Freedom Systems

Newton's Second Law

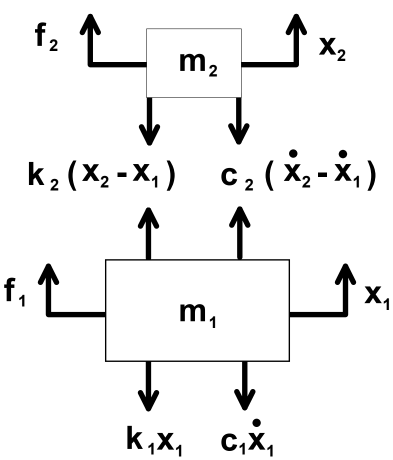
$$m_1\ddot{x}_1 = f_1(t) - c_1\dot{x}_1 + c_2(\dot{x}_2 - \dot{x}_1) - k_1x_1 + k_2(x_2 - x_1)$$

$$m_2\ddot{x}_2 = f_2(t) - c_2(\dot{x}_2 - \dot{x}_1) - k_2(x_2 - x_1)$$

Rearrange terms

$$m_1\ddot{x}_1 + (c_1 + c_2)\dot{x}_1 - c_2\dot{x}_2 + (k_1 + k_2)x_1 - k_2x_2 = f_1(t)$$

$$m_2\ddot{x}_2 - c_2\dot{x}_1 + c_2\dot{x}_2 - k_2x_1 + k_2x_2 = f_2(t)$$



Multiple Degree of Freedom Systems

Matrix Formulation

$$\begin{bmatrix} m_1 & \\ & m_2 \end{bmatrix} \begin{Bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{Bmatrix} + \begin{bmatrix} (c_1 + c_2) & -c_2 \\ -c_2 & c_2 \end{bmatrix} \begin{Bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{Bmatrix} + \begin{bmatrix} (k_1 + k_2) & -k_2 \\ -k_2 & k_2 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} = \begin{Bmatrix} f_1(t) \\ f_2(t) \end{Bmatrix}$$

Matrices and Linear Algebra are important !!!

MDOF - Frequencies and Mode Shapes

Example 5.1.1

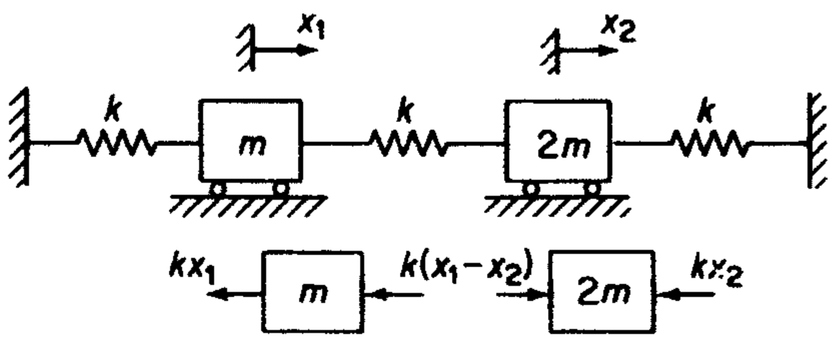


FIGURE 5.1.1.

$$\begin{aligned} m\ddot{x}_1 &= -kx_1 + k(x_2 - x_1) \\ 2m\ddot{x}_2 &= -k(x_2 - x_1) - kx_2 \end{aligned} \qquad (5.1.1)$$

MDOF - Frequencies and Mode Shapes

For normal mode type of oscillation, we can write

$$x_1 = A_1 \sin \omega t \quad \text{or} \quad A_1 e^{i\omega t}$$
$$x_2 = A_2 \sin \omega t \quad \text{or} \quad A_2 e^{i\omega t} \tag{5.1.2}$$

substituting into the differential equation yields

$$\left(2k - \omega^2 m\right) A_1 - k A_2 = 0$$
$$-k A_1 + \left(2k - 2\omega^2 m\right) A_2 = 0 \tag{5.1.3}$$

MDOF - Frequencies and Mode Shapes

In matrix form this is

$$\begin{bmatrix} \left(2k-\omega^2 m\right) & -k \\ -k & \left(2k-2\omega^2 m\right) \end{bmatrix} \begin{Bmatrix} A_1 \\ A_2 \end{Bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \qquad (5.1.4)$$

and the determinant of the matrix is

$$\begin{vmatrix} \left(2k-\omega^2 m\right) & -k \\ -k & \left(2k-2\omega^2 m\right) \end{vmatrix} = 0 \qquad (5.1.5)$$

whose solution yields the eigenvalues

$$\omega^4 - \frac{3k}{m}\omega^2 + \frac{3}{2}\left(\frac{k}{m}\right)^2 = \lambda^2 - \frac{3k}{m}\lambda + \frac{3}{2}\left(\frac{k}{m}\right)^2 = 0 \qquad (5.1.6)$$

MDOF - Frequencies and Mode Shapes

The frequencies of the system are

$$\lambda_1 = 0.634\frac{k}{m} \quad \Rightarrow \quad \omega_1 = \sqrt{0.634\frac{k}{m}}$$
$$\lambda_2 = 2.366\frac{k}{m} \quad \Rightarrow \quad \omega_2 = \sqrt{2.366\frac{k}{m}} \qquad (5.1.7)$$

and the general ratio of response is

$$\frac{A_1}{A_2} = \frac{k}{2k - \omega^2 m} = \frac{2k - 2\omega^2 m}{k} \qquad (5.1.8)$$

MDOF - Frequencies and Mode Shapes

The ratio for the first frequency , ω_1, is

$$\left(\frac{A_1}{A_2}\right)^{(1)} = \frac{k}{2k - \omega^2 m} = 0.731 \qquad (5.1.8a)$$

The ratio for the second frequency, ω_2, is

$$\left(\frac{A_1}{A_2}\right)^{(2)} = \frac{k}{2k - \omega^2 m} = -2.73 \qquad (5.1.8b)$$

MDOF - Frequencies and Mode Shapes

The mode shape for the two different modes is

$$\phi_1(x) = \begin{Bmatrix} 0.731 \\ 1 \end{Bmatrix} \quad ; \quad \phi_2(x) = \begin{Bmatrix} -2.73 \\ 1 \end{Bmatrix} \tag{5.1.8}$$

Each mode oscillates according to

$$\begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix}^{(1)} = A_1 \begin{Bmatrix} 0.731 \\ 1 \end{Bmatrix} \sin(\omega_1 t + \psi_1)$$

$$\begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix}^{(2)} = A_1 \begin{Bmatrix} -2.73 \\ 1 \end{Bmatrix} \sin(\omega_2 t + \psi_2)$$

(5.1.8b

MDOF - Initial Conditions

The general description of the system (for the example considered) is

$$\omega_1 = \sqrt{0.634 \frac{k}{m}} \quad ; \quad \phi_1(x) = \begin{Bmatrix} 0.731 \\ 1 \end{Bmatrix}$$

$$\omega_2 = \sqrt{2.366 \frac{k}{m}} \quad ; \quad \phi_2(x) = \begin{Bmatrix} -2.73 \\ 1 \end{Bmatrix}$$

and the initial conditions are specified as

$$\begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix}^{(i)} = c_i \phi_i \sin\left(\omega_i t + \psi_i\right) \qquad i = 1,2 \qquad (5.2.1)$$

MDOF - Initial Conditions

The displacement is written as

$$\begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} = c_1 \begin{Bmatrix} 0.731 \\ 1 \end{Bmatrix} \sin(\omega_1 t + \psi_1) + c_2 \begin{Bmatrix} -2.732 \\ 1 \end{Bmatrix} \sin(\omega_2 t + \psi_2) \tag{5.2.2}$$

The velocity is written as

$$\begin{Bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{Bmatrix} = \omega_1 c_1 \begin{Bmatrix} 0.731 \\ 1 \end{Bmatrix} \cos(\omega_1 t + \psi_1) + \omega_2 c_2 \begin{Bmatrix} -2.732 \\ 1 \end{Bmatrix} \cos(\omega_2 t + \psi_2) \tag{5.2.3}$$

MDOF - Initial Conditions

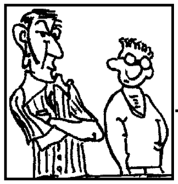
Example 5.2.1 - Initial conditions are:

$$\begin{Bmatrix} x_1(0) \\ x_2(0) \end{Bmatrix} = \begin{Bmatrix} 2.0 \\ 4.0 \end{Bmatrix} \quad ; \quad \begin{Bmatrix} \dot{x}_1(0) \\ \dot{x}_2(0) \end{Bmatrix} = \begin{Bmatrix} 0.0 \\ 0.0 \end{Bmatrix}$$

which correspond to

$$\begin{Bmatrix} 2 \\ 4 \end{Bmatrix} = c_1 \begin{Bmatrix} 0.731 \\ 1 \end{Bmatrix} \sin(\psi_1) + c_2 \begin{Bmatrix} -2.732 \\ 1 \end{Bmatrix} \sin(\psi_2) \qquad (5.2.2a)$$

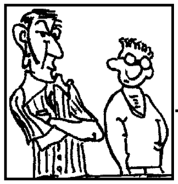
$$\begin{Bmatrix} 0 \\ 0 \end{Bmatrix} = \omega_1 c_1 \begin{Bmatrix} 0.731 \\ 1 \end{Bmatrix} \cos(\psi_1) + \omega_2 c_2 \begin{Bmatrix} -2.732 \\ 1 \end{Bmatrix} \cos(\psi_2) \qquad (5.2.3a)$$



MDOF - Initial Conditions

Example 5.2.1 - upon solving these equations for the response due to the specified initial conditions yields:

$$\begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} = 3.732 \begin{Bmatrix} 0.731 \\ 1 \end{Bmatrix} \cos(\omega_1 t) + 0.268 \begin{Bmatrix} -2.732 \\ 1 \end{Bmatrix} \cos(\omega_2 t)$$



MDOF - Coordinate Coupling

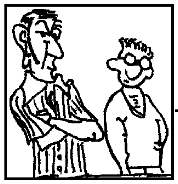
Coordinate coupling exists in many problems. Either static coupling, dynamic coupling or both static and dynamic coupling can exist.

The equations of motion are:

$$\begin{aligned} m_{11}\ddot{x}_1 + m_{12}\ddot{x}_2 + k_{11}x_1 + k_{12}x_2 &= 0 \\ m_{21}\ddot{x}_1 + m_{22}\ddot{x}_2 + k_{21}x_1 + k_{22}x_2 &= 0 \end{aligned} \qquad (5.3.1)$$

and can be cast in matrix form as:

$$\begin{bmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{bmatrix} \begin{Bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{Bmatrix} + \begin{bmatrix} k_{11} & k_{12} \\ k_{21} & k_{22} \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} \qquad (5.3.2)$$



MDOF - Coordinate Coupling

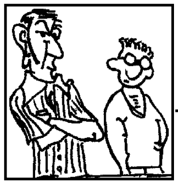
Coordinate coupling can be eliminated through a transformation to a different coordinate system wherein the independent variables are not coupling either statically or dynamically.

These coordinates are referred to as

'principal coordinates'

or

'normal coordinates'



MDOF - Coordinate Coupling

For systems with general damping, this is not easily possible unless the damping is of a special form or the system is first converted to the state space formulation of the system equations

$$\begin{bmatrix} m_{11} & 0 \\ 0 & m_{22} \end{bmatrix} \begin{Bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{Bmatrix} + \begin{bmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{bmatrix} \begin{Bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{Bmatrix} + \begin{bmatrix} k_{11} & 0 \\ 0 & k_{22} \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} \quad (5.3.3)$$

MDOF - Coordinate Coupling

Example 5.3.1 Static Coupling

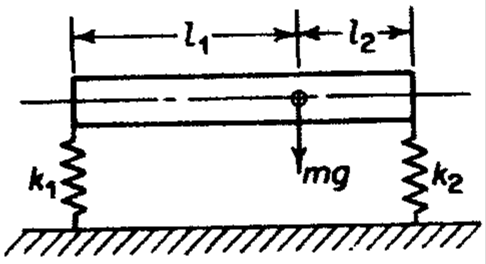


FIGURE 5.3.1.

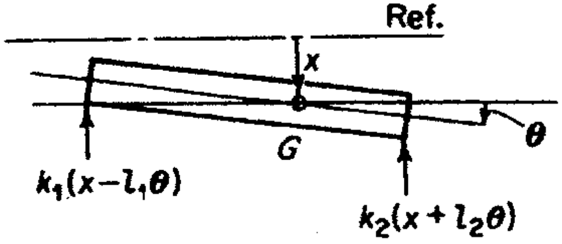
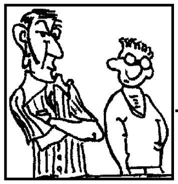


FIGURE 5.3.2. Coordinates leading to static coupling.

$$\begin{bmatrix} m & 0 \\ 0 & J \end{bmatrix} \begin{Bmatrix} \ddot{x} \\ \ddot{\theta} \end{Bmatrix} + \begin{bmatrix} (k_1 + k_2) & (k_2 l_2 - k_1 l_1) \\ (k_2 l_2 - k_1 l_1) & (k_1 l_1^2 + k_2 l_2^2) \end{bmatrix} \begin{Bmatrix} x \\ \theta \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$



MDOF - Coordinate Coupling

Example 5.3.1 Dynamic Coupling

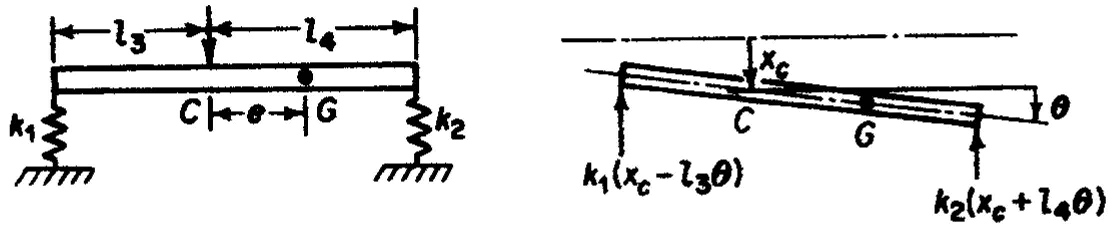


FIGURE 5.3.3. Coordinates leading to dynamic coupling.

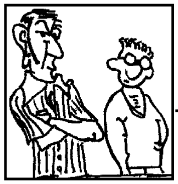
$$\begin{bmatrix} m & me \\ me & J_c \end{bmatrix} \begin{Bmatrix} \ddot{x}_c \\ \ddot{\theta} \end{Bmatrix} + \begin{bmatrix} (k_1 + k_2) & 0 \\ 0 & (k_1 l_3^2 + k_2 l_4^2) \end{bmatrix} \begin{Bmatrix} x_c \\ \theta \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

MDOF - Coordinate Coupling

Example 5.3.1 Static & Dynamic Coupling

FIGURE 5.3.4. Coordinates leading to static and dynamic coupling.

$$\begin{bmatrix} m & ml_1 \\ ml_1 & J \end{bmatrix} \begin{Bmatrix} \ddot{x}_1 \\ \ddot{\theta} \end{Bmatrix} + \begin{bmatrix} (k_1 + k_2) & k_2 l \\ k_2 l & k_2 l^2 \end{bmatrix} \begin{Bmatrix} x_1 \\ \theta \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$



MDOF - Forced Harmonic Vibration

Consider a system excited by a harmonic force

$$\begin{bmatrix} m_{11} & \\ & m_{22} \end{bmatrix} \begin{Bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{Bmatrix} + \begin{bmatrix} k_{11} & k_{12} \\ k_{21} & k_{22} \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} = \begin{Bmatrix} F_1 \\ 0 \end{Bmatrix} \sin \omega t \qquad (5.4.1)$$

which has a solution assumed to be

$$\begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} = \begin{Bmatrix} X_1 \\ X_2 \end{Bmatrix} \sin \omega t$$

MDOF - Forced Harmonic Vibration

Substituting into the differential equation yields

$$\begin{bmatrix} (k_{11}-m_{11}\omega^2) & k_{12} \\ k_{21} & (k_{22}-m_{22}\omega^2) \end{bmatrix} \begin{Bmatrix} X_1 \\ X_2 \end{Bmatrix} = \begin{Bmatrix} F_1 \\ 0 \end{Bmatrix} \qquad (5.4.2)$$

which is generally written in terms of the impedance matrix as

$$[Z(\omega)] \begin{Bmatrix} X_1 \\ X_2 \end{Bmatrix} = \begin{Bmatrix} F_1 \\ 0 \end{Bmatrix}$$

MDOF - Forced Harmonic Vibration

Solving this yields

$$\begin{Bmatrix} X_1 \\ X_2 \end{Bmatrix} = [Z(\omega)]^{-1} \begin{Bmatrix} F_1 \\ 0 \end{Bmatrix} = \frac{\mathrm{Adj}[Z(\omega)]}{\det Z(\omega)} \begin{Bmatrix} F_1 \\ 0 \end{Bmatrix} \qquad (5.4.3)$$

where the adjoint matrix and determinant are

$$\mathrm{Adj}[Z(\omega)] = \begin{bmatrix} \left(k_{22} - m_{22}\omega^2\right) & -k_{12} \\ -k_{21} & \left(k_{11} - m_{11}\omega^2\right) \end{bmatrix}$$

$$\det Z(\omega) = m_1 m_2 \left(\omega_1^2 - \omega^2\right)\left(\omega_2^2 - \omega^2\right)$$

MDOF - Forced Harmonic Vibration

The general equation becomes

$$\begin{Bmatrix} X_1 \\ X_2 \end{Bmatrix} = \frac{\begin{bmatrix} \left(k_{22} - m_{22}\omega^2\right) & -k_{12} \\ -k_{21} & \left(k_{11} - m_{11}\omega^2\right) \end{bmatrix}}{m_1 m_2 \left(\omega_1^2 - \omega^2\right)\left(\omega_2^2 - \omega^2\right)} \begin{Bmatrix} F_1 \\ 0 \end{Bmatrix} \qquad (5.4.3)$$

and the amplitudes of response are

$$X_1 = \frac{\left(k_{22} - m_{22}\omega^2\right) F_1}{m_1 m_2 \left(\omega_1^2 - \omega^2\right)\left(\omega_2^2 - \omega^2\right)}$$

$$X_2 = \frac{-k_{21} F_1}{m_1 m_2 \left(\omega_1^2 - \omega^2\right)\left(\omega_2^2 - \omega^2\right)} \qquad (5.4.6)$$

Example 5.4.1 and 5.4.2 are good examples

MDOF - Vibration Absorber

A very common, practical application of a 2 DOF system is that of the 'tuned absorber'. This is commonly used to minimize objectionable resonance

Recall

$$\omega_1^2 = \frac{k_1}{m_1} \quad ; \quad \omega_2^2 = \frac{k_2}{m_2} \tag{5.6.1}$$

The amplitude of response for X_1 is

$$\frac{X_1 k_1}{F_0} = \frac{\left[1 - \left(\frac{\omega}{\omega_2}\right)^2\right]}{\left[1 + \frac{k_2}{k_1} - \left(\frac{\omega}{\omega_1}\right)^2\right]\left[1 - \left(\frac{\omega}{\omega_2}\right)^2\right] - \frac{k_2}{k_1}} \tag{5.3.2}$$

MDOF - Tuned Absorber

$$\frac{X_1k_1}{F_0}=\frac{\left[1-\left(\frac{\omega}{\omega_2}\right)^2\right]}{\left[1+\frac{k_2}{k_1}-\left(\frac{\omega}{\omega_1}\right)^2\right]\left[1-\left(\frac{\omega}{\omega_2}\right)^2\right]-\frac{k_2}{k_1}}$$

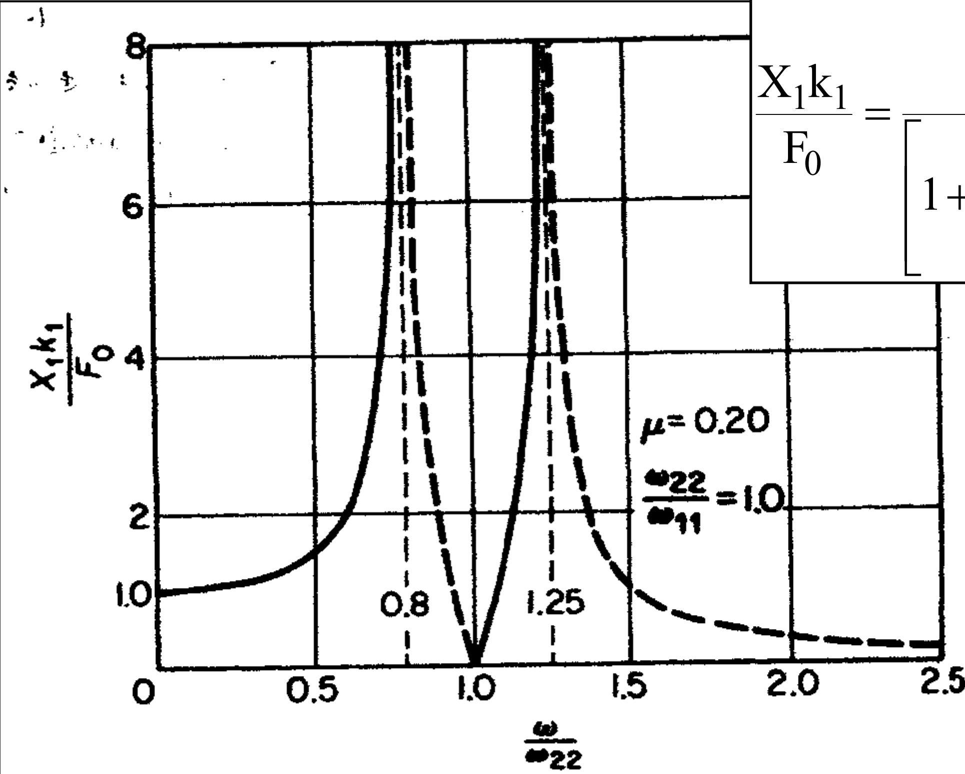


FIGURE 5.6.2. Response vs. frequency.

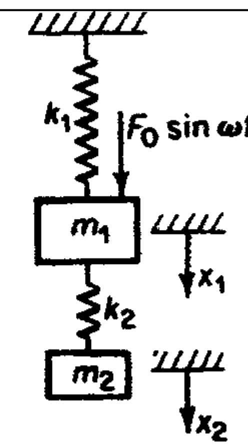


FIGURE 5.6.1. Vibration absorber.

MDOF - Tuned Absorber

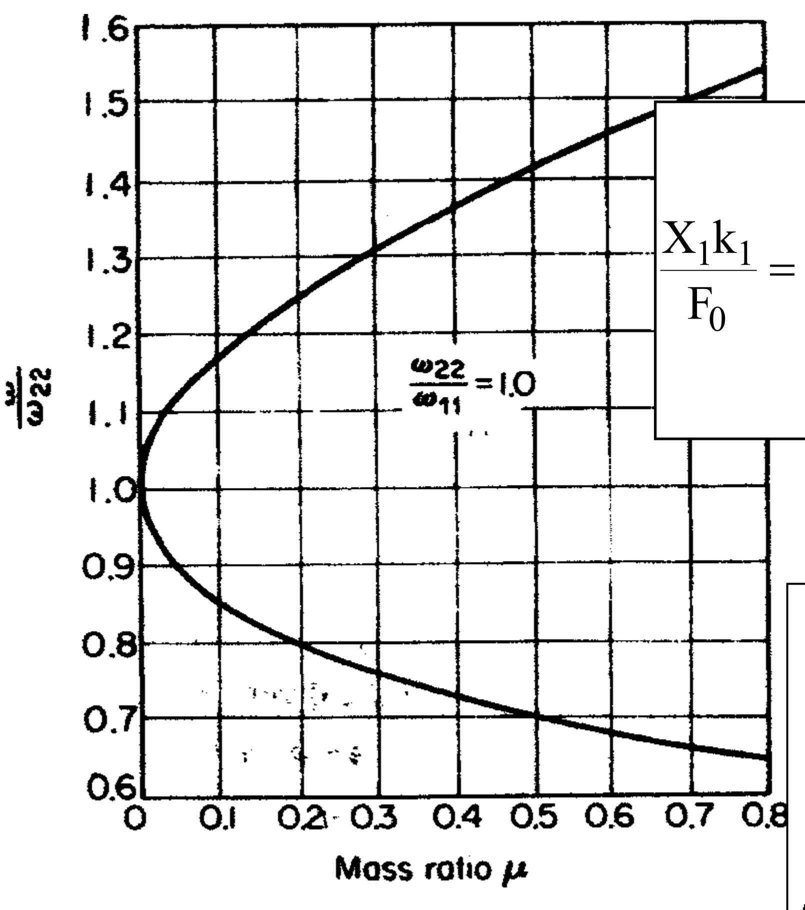


FIGURE 5.6.3 Natural frequencies vs. μ vs. m_2/m_1.

$$\frac{X_1 k_1}{F_0} = \frac{\left[1 - \left(\frac{\omega}{\omega_2}\right)^2\right]}{\left[1 + \frac{k_2}{k_1} - \left(\frac{\omega}{\omega_1}\right)^2\right]\left[1 - \left(\frac{\omega}{\omega_2}\right)^2\right] - \frac{k_2}{k_1}}$$

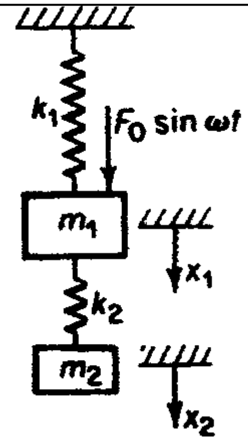
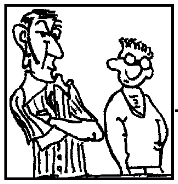


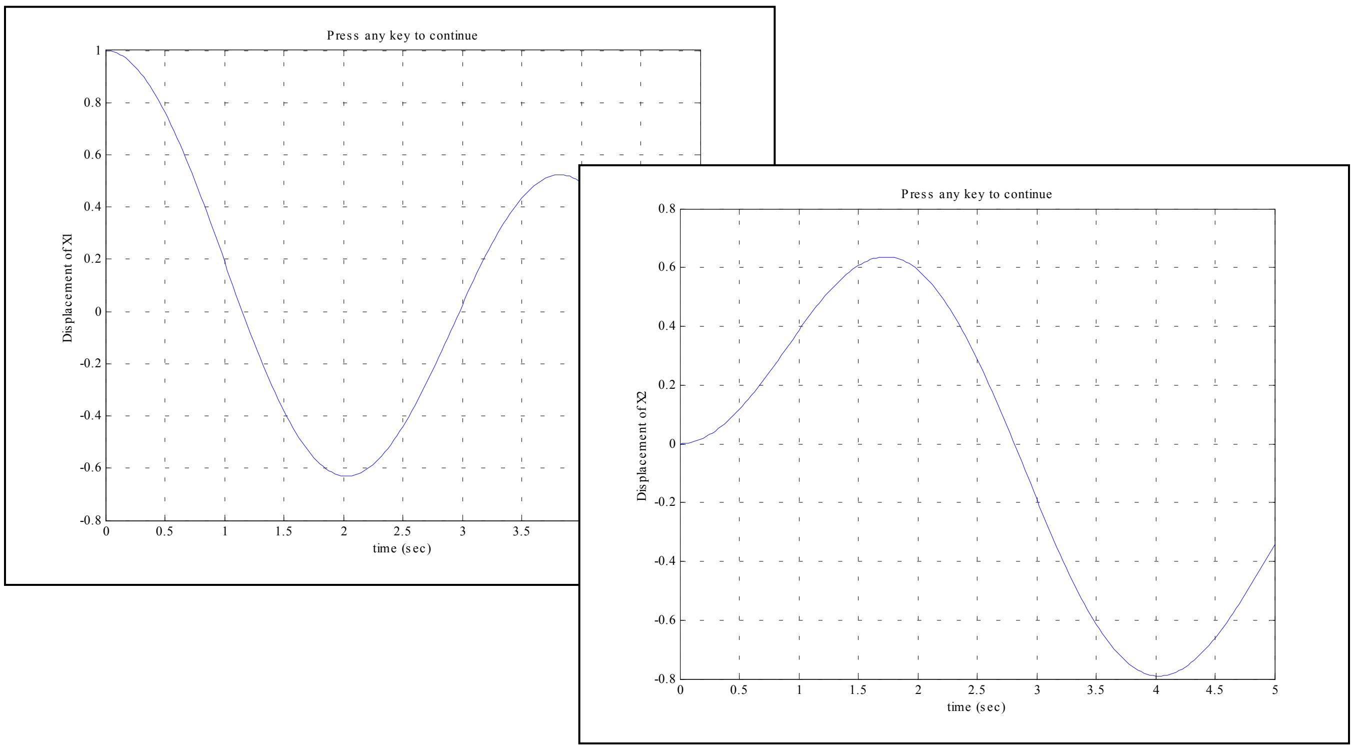
FIGURE 5.6.1. Vibration absorber.

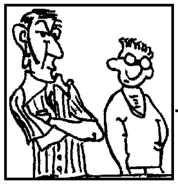


MATLAB Examples - VTB4_2

VIBRATION TOOLBOX EXAMPLE 4_2

```
>> clear
>> m=[1 0;0 1];k=[2 -1;-1 1];x0=[1;0];v0=[0;0];tf=5;plotpar=1;
>> [x,v,t]=VTB4_2(m,k,x0,v0,tf,plotpar);
```

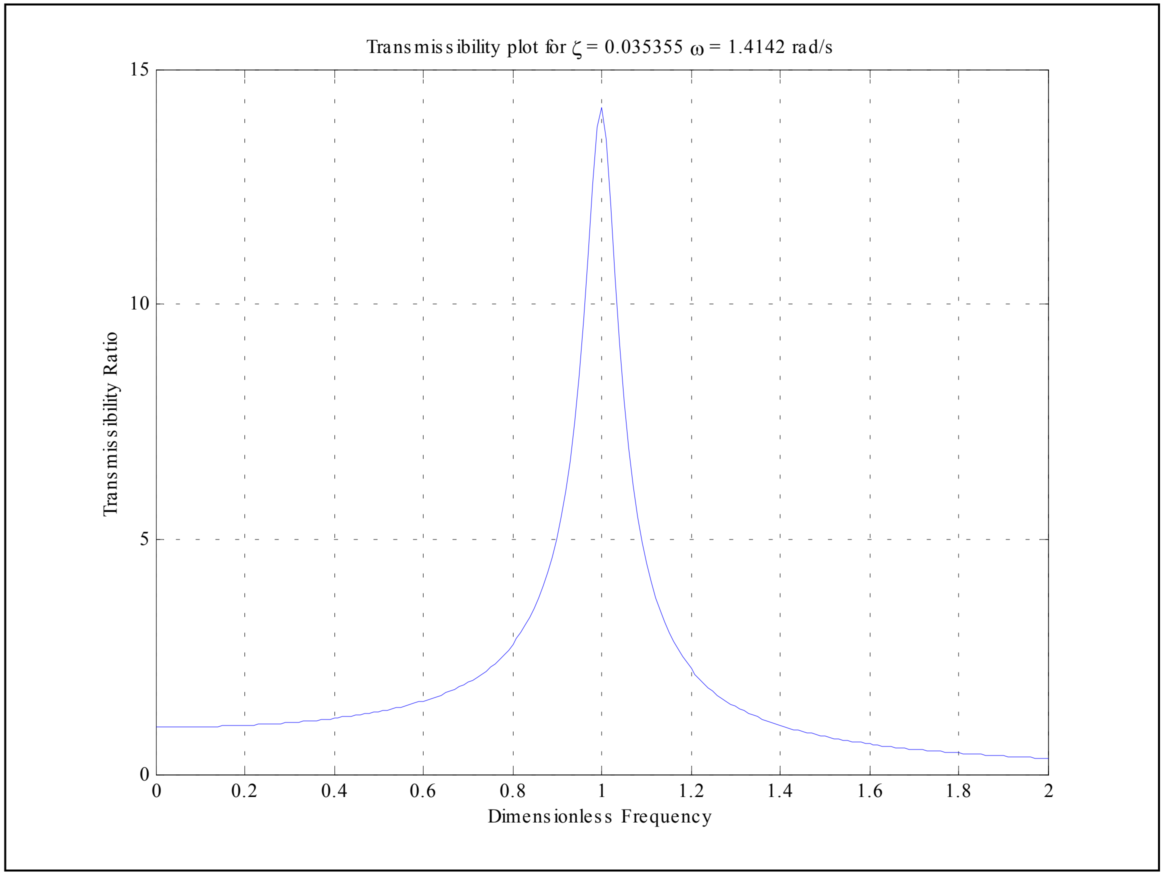


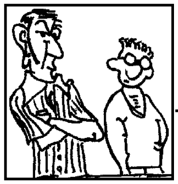


MATLAB Examples - VTB5_1

VIBRATION TOOLBOX EXAMPLE 5_1

```
>> clear
>> m=1;   c=.1;   k=2;
>> VTB5_1(m,c,k)
>>
```





MATLAB Examples - VTB5_4

VIBRATION TOOLBOX EXAMPLE 5_4

```
>> beta=1

beta =

     1

>> VTB5_4(beta)
>>
```