Step Response of Second-Order Systems

INTRODUCTION

This document discusses the response of a second-order system, such as the mass-spring-dashpot shown in Fig. 1, to a step function. The modeling of a step response in MATLAB and SIMULINK will also be discussed.

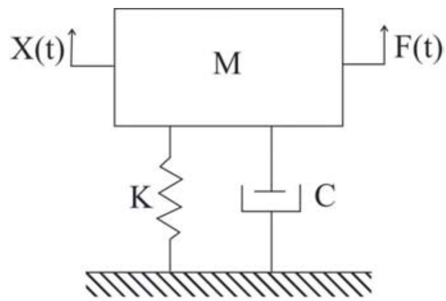

Fig. 1. Single-degree-of-freedom mass-spring-dashpot system.

For more background on second-order systems in general, see the tutorial on second-order system theory.

STEP FUNCTION

Mathematically, a unit step function can be described by

$$f(t) = \begin{cases} 0 & \text{for } t \leq 0 \\ 1 & \text{for } t > 0 \end{cases}. \tag{1}$$

Essentially, it is a function which jumps from zero to 1 at time t = 0. This can be physically related to a unit force being applied instantaneously to a structure. A unit step function is generally denoted by u(t), and is shown graphically in Fig. 2.

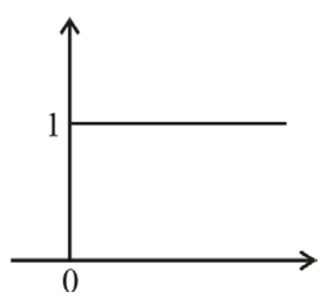

Fig. 2. Unit step function.

General step functions can have any height, or be applied at times other than zero.

In order to determine the response of a dynamic system to a step function, it is convenient to use Laplace Transform. The Laplace Transform of a unit step function is

$$L\{u(t)\} = \frac{1}{s}. \qquad (2)$$

EQUATIONS DESCRIBING SYSTEM RESPONSE

The general equation of motion for a second-order system with an applied unit step function is

$$\ddot{x} + 2\zeta\omega_n \dot{x} + \omega_n^2 x = u(t). \qquad (3)$$

The form of the response of the system depends on whether the system is under-damped, critically damped, or over-damped.

Under-Damped

The response of an under-damped second-order-system (ζ<1) to a unit step input, assuming zero initial conditions, is

$$x(t) = \frac{1}{\omega_n^2}\left[1 - e^{-\sigma t}\left(\cos\omega_d t + \frac{\zeta}{\sqrt{1-\zeta^2}}\sin\omega_d t\right)\right]. \qquad (4)$$

Critically Damped

The unit step response of a critically damped system (ζ=1) with zero initial conditions is given by

$$x(t) = \frac{1}{\omega_n^2}\left[1 - e^{-\omega_n t}(1 - \omega_n t)\right]. \qquad (5)$$

Over-Damped

The response of an over-damped system (ζ>1), again assuming zero initial conditions, is

$$x(t) = \frac{1}{\omega_n^2}\left[1 + \frac{1}{2\sqrt{\zeta^2-1}}\left(\frac{1}{-\zeta+\sqrt{\zeta^2-1}}e^{-\omega_n\left(\zeta-\sqrt{\zeta^2-1}\right)t} + \frac{1}{\zeta+\sqrt{\zeta^2-1}}e^{-\omega_n\left(\zeta+\sqrt{\zeta^2-1}\right)t}\right)\right]. \qquad (6)$$

FORM OF SYSTEM RESPONSE

Fig. 3 shows the unit step response of a under-damped, critically damped, and over-damped system.

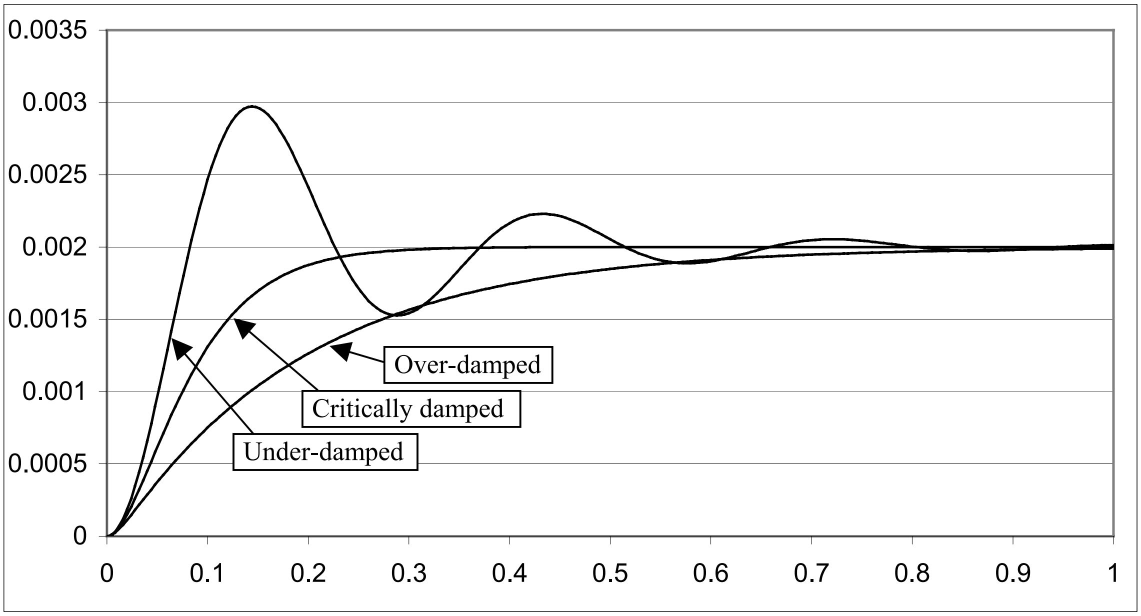

Fig. 3. Step response for under-damped, critically damped, and over-damped systems.

Table 1 gives the properties of the three systems.

Table 1. Damping ratios for three example systems.

System type	Mass	Stiffness	Damping	Damping ratio (ζ)
Under-damped	1	500	10	0.22
Critically damped			44.7	1.0
Over-damped			100	2.2

In Fig. 3, note that the steady-state response is at 0.002, despite the steady state of the forcing function being at 1. The reason for this is clear when the static response is considered. For a simple mass-spring system, the displacement is given by

$$x = \frac{F}{k} \tag{7}$$

where F is the applied force and k is the spring stiffness. As the damping only applies a force when the system is in motion, (7) can be used when considering the static displacement of the system. Clearly, the displacement of the system is scaled by the spring stiffness. Therefore, for a unit load on a system with a spring stiffness of 500, the static displacement would be

$$\frac{1\mathrm{N}}{500\,{}^{\mathrm{N}}\!/_{\mathrm{m}}} = 0.002\mathrm{m}\,. \tag{8}$$

Despite a unit force being applied, the steady state deflection is only 1 if k = 1.

CALCULATING RESPONSE IN SIMULINK

A simple Simulink model, like that shown in Fig. 4, can be used to find the step response of a second-order system.

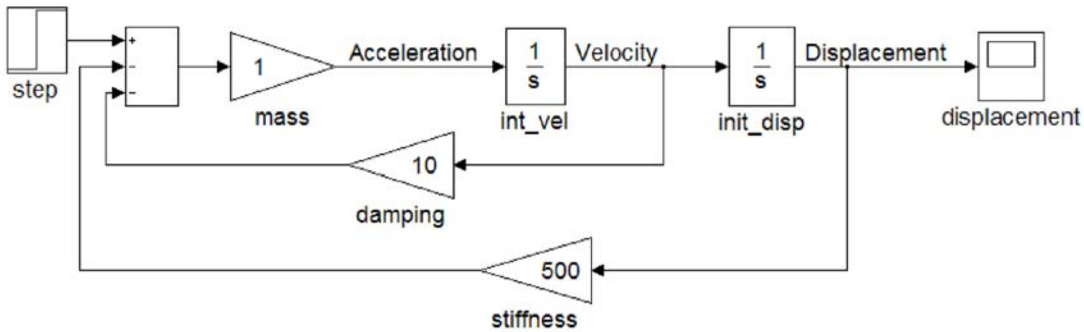

Fig. 4. Simulink model to determine step response.

Construct the model shown in Fig. 4, using the values given in Table 2. These are the same values as for the system discussed above, whose response is given in Fig. 3. In this example, and the one using MATLAB, the value for damping used produces an underdamped system. Try varying the damping value and seeing the change in the response.

Table 2. Values used in Simulink model.

Step block	Step time	0
	Initial value	0
	Final value	1
Mass gain block	Gain	1
Damping gain block	Gain	10
Stiffness gain block	Gain	500
Integration blocks (both)	Initial condition	0

Double-click on the displacement scope block to bring up the plot of the system response.

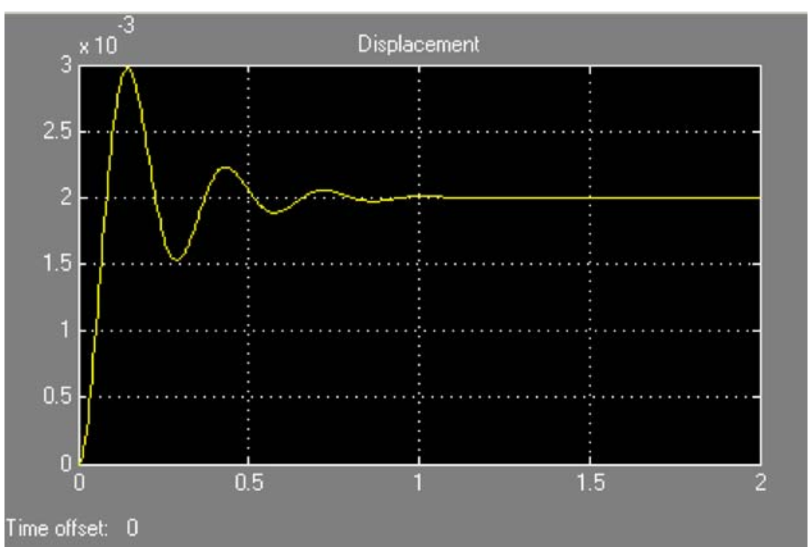

Fig. 5. System response found from Simulink model.

CALCULATING RESPONSE IN MATLAB

MATLAB can also be used to see the step response of a second-order system. The code shown below was used to produce the plot shown in Fig. 6.

```
% identify the characteristics of the system
% (values shown are for the underdamped case)
m = 1;
c = 10;
k = 500;

% identify the numerator and the denominator of the system transfer
% function
num = [0 0 1];
den = [m c k];

% plot the unit step response
step(num,den)
```

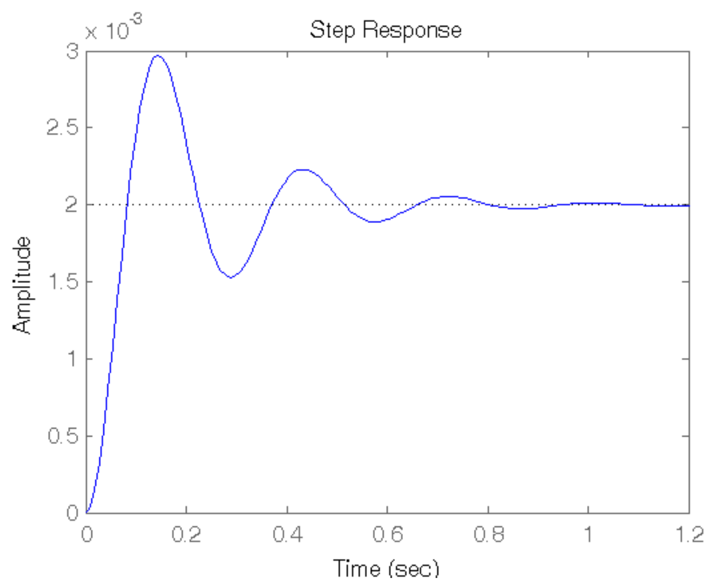

Fig. 6. Unit step response found in MATLAB.