Theory of Second-Order Systems

INTRODUCTION

A second-order dynamic system is one whose response can be described by a second-order ordinary differential equation (ODE). A second-order ODE is one in which the highest-order derivative is a second derivative.

Many mechanical systems can be modeled as second-order systems. A commonly used example of such a system is the simple single-degree-of-freedom mass-spring-dashpot system shown in Fig. 1. This model will be used as an example in this and other tutorials in this series.

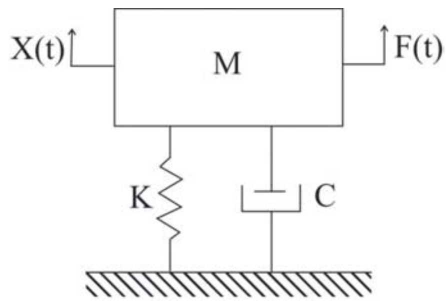

Fig. 1. Mass-spring-dashpot system.

Many electrical circuits, such as the RLC circuit shown in Fig. 2, can also be modeled as second-order systems.

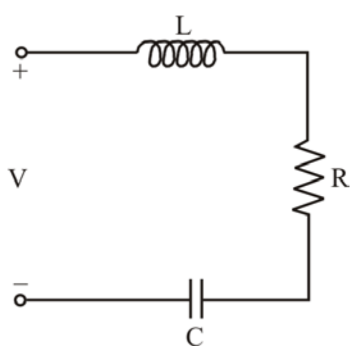

Fig. 2. Second-order RLC circuit

SYSTEM EQUATION

In standard form, the ODE describing the behavior of a second-order dynamic system is

$$\ddot{x} + 2\zeta\omega_n \dot{x} + \omega_n^2 x = f(t), \quad (1)$$

where

x = displacement, or the equivalent property for the given system,

$\dot{x}$ = velocity, or equivalent,

$\ddot{x}$ = acceleration, or equivalent,

ω_n = the natural frequency of the system,

ζ = the damping ratio, and

f(t) = the forcing function, a function of time.

The natural frequency of the system is the frequency at which it will oscillate if set into motion and allowed to move freely.

The damping ratio is defined as

$$\zeta = \frac{c}{c_c}, \tag{2}$$

where

c = the damping in the system, and

c_c = the critical damping.

Damping is a measure of how fast the system dissipates energy. Critical damping is the amount of damping for a particular system which will cause it to reach the steady-state response in the minimum possible time.

The function on the right-hand side of (1), f(t), is the forcing function—some input to the system which is driving its response.

MASS-SPRING-DASHPOT SYSTEM

For the mass-spring-dashpot system shown in Fig. 1, the equation of motion is

$$m\ddot{x} + c\dot{x} + kx = f(t), \tag{3}$$

where

m = effective mass of system,

c = damping,

k = stiffness, and

f(t) = the forcing function.

When this equation is put into standard form and compared to (1), it can easily be seen that

$$\omega_n = \sqrt{\frac{k}{m}}, \tag{4}$$

and

$$\zeta = \frac{c}{2m\omega_n}. \tag{5}$$

By comparing (5) and (2), it can be seen that, for the mass-spring-dashpot system,

$$c_c = 2m\omega_n. \tag{6}$$

SYSTEM RESPONSE

In order to determine the actual response, such as the displacement, of a second order system, the differential equation must be solved. The ODE has a homogeneous solution and a particular solution, x_h and x_p, which describe the response of the system.

Homogeneous Solution

The homogeneous solution depends on the inherent characteristics of the system and describes the system's free response. It describes how the system will respond if set into motion, such as with an initial displacement or initial velocity, and then allowed to move freely. The free response is the solution of the equation

$$\ddot{x} + 2\zeta\omega_n \dot{x} + \omega_n^2 x = 0 , \tag{7}$$

which is identical to (1) except that there is no forcing function; the right-hand side of the equation is equal to zero. If it is now assumed, as it was for the first-order system, that the solution is in the form

$$x(t) = e^{\lambda t} , \tag{8}$$

then the characteristic equation is found to be

$$\lambda^2 + 2\zeta\omega_n \lambda + \omega_n^2 = 0 . \tag{9}$$

Solving for λ, it is found that

$$\lambda = -\zeta\omega_n \pm \omega_n \sqrt{\zeta^2 - 1} \tag{10}$$

Clearly, the form of the solution depends strongly on whether the quantity under the radical is positive, negative, or zero.

Effect of Damping Ratio on System Response

Depending on whether the quantity $(\zeta^2 - 1)$ is negative, zero, or positive, the system is underdamped, critically damped, or overdamped, respectively.

Underdamped: When this quantity is negative ($\zeta<1$), the system is said to be underdamped. This is, by far, the most common case for structural systems. When a system is underdamped, it will oscillate around the steady state condition before leveling out at steady state. This can clearly be seen in Fig. 3. For an underdamped system, the solution to the equation of motion is

$$x_p(t) = e^{-\sigma t}\left[x_0 \cos\omega_d t + \left(\frac{\sigma x_0 + v_0}{\omega_d} \right) \sin\omega_d t \right] \tag{11}$$

where

$$\sigma = \zeta\omega_n , \tag{12}$$

and

$$\omega_d = \omega_n \sqrt{1 - \zeta^2} . \tag{13}$$

The value ω_d is the damped natural frequency. This is the frequency at which the system will oscillate when there is damping present. If measurements are being taken on an actual dynamic system, the frequency being measured is the damped natural frequency. For very small amounts of damping, however, $\omega_d \cong \omega_n$.

Note that the solution in (11) allows for the possibility that the system is subjected to initial conditions—x_0 and v_0 are initial displacement and initial velocity, respectively.

Critically damped: When the quantity under the radical is zero ($\zeta=1$), the system is critically damped. As explained above, a critically damped system will reach the steady-state response in the minimum possible time. For a critically damped system, the solution to the equation of motion is

$$x_p(t) = x_0 e^{-\omega_n t} + (\omega_n x_0 + v_0) t e^{-\omega_n t} \tag{14}$$

Overdamped: When the quantity under the radical in (10) is positive (ζ>1), the system is overdamped. This means that it has greater than critical damping; the response will lag behind the input. The greater the damping beyond critical damping, the more slowly the system will respond. For an overdamped system, the solution to the equation of motion is

$$x_p(t) = \left[\frac{\omega_n\left(\zeta + \sqrt{\zeta^2 - 1}\right)x_0 + v_0}{2\omega_n\sqrt{\zeta^2 - 1}}\right] e^{-\omega_n\left(\zeta - \sqrt{\zeta^2 - 1}\right)t} - \left[\frac{\omega_n\left(\zeta - \sqrt{\zeta^2 - 1}\right)x_0 + v_0}{2\omega_n\sqrt{\zeta^2 - 1}}\right] e^{-\omega_n\left(\zeta + \sqrt{\zeta^2 - 1}\right)t}. \tag{15}$$

These three cases are compared in Fig. 3, which shows the response of a second-order system to an initial displacement of 0.01. Note that this is the free response of the system, because no force acts on the system after time t = 0.

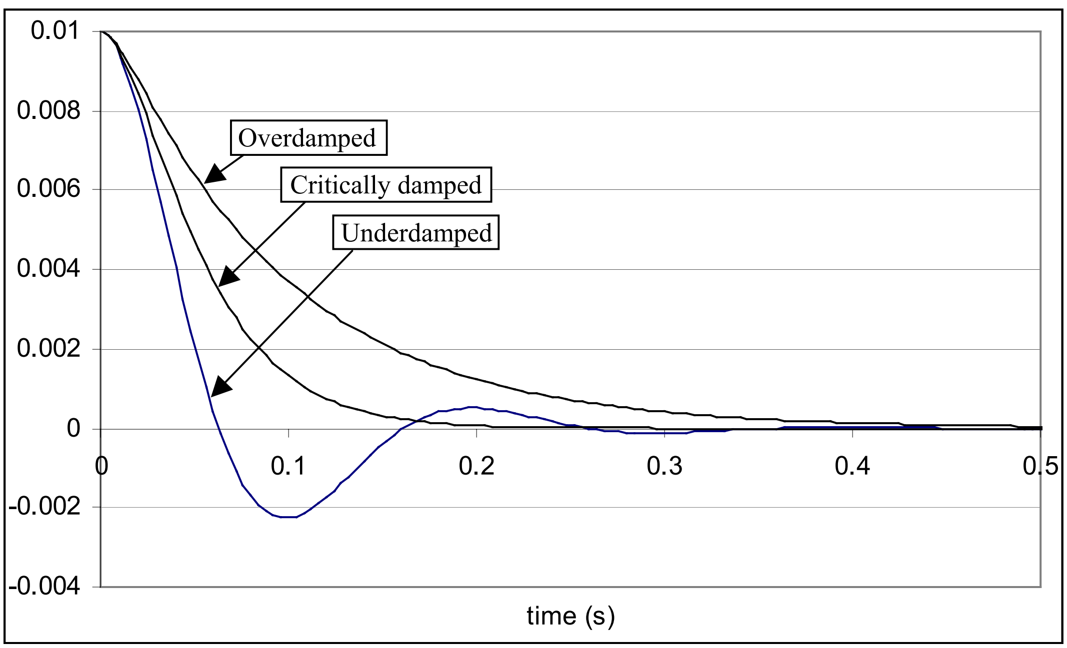

Fig. 3. Comparison of underdamped, critically damped, and overdamped cases.

Particular Solution

The particular solution to the ODE depends on the inputs to the system. It describes the response of the system when the right-hand side of (1), f(t), is non-zero. The form of the solution depends on the actual forcing function, which can be any time-varying function. Typical functions which are found are impulses and step functions. The response of a second-order system to these forcing functions is discussed in detail in separate documents.