Q factor
Nick Lepeshkin

1 Damped oscillations

The equation of motion for a damped oscillator

$$\frac{d^2 x}{dt^2} = -\omega_0^2 x - \frac{b}{m} \frac{dx}{dt}$$

leads to the following solution $x(t)$

$$x(t) = A_0 e^{-\frac{t}{\tau}} \cos(\omega t + \phi_0)$$

where $A_0$ is the initial amplitude, $\tau \equiv m/b$ is the decay time, and frequency $\omega = \sqrt{\omega_0^2 - \left(\frac{1}{\tau^2}\right)^2}$. The quality factor $Q$ defined as

$$Q \equiv 2\pi \frac{\tau}{T} = \omega \tau$$

is a convenient measure of how much longer the decay time is compared to the period. Lightly damped oscillations are referred to as high $Q$, and heavier damped oscillations - as low $Q$. Informally, the quality factor represents the number of cycles completed by the oscillator before it ”rings down” or ”runs out of energy”. More rigorously, for $Q \gg 1$

$$Q = \pi \frac{A}{\Delta A}$$

where $\Delta A$ is the change in amplitude over one period (cycle), or, in terms of energy

$$Q = 2\pi \frac{E}{\Delta E}$$

where $\Delta E$ is the decrease in mechanical energy over one period (cycle). Try and convince yourself that all of the expression for $Q$ given above are consistent with each other. Keep in mind, that $e^x \approx 1 + x$ for $x \ll 1$. The frequency of the damped oscillation could also be recast in terms of $Q$ as follows

$$\omega \approx \omega_0 \left(1 - \frac{1}{8Q^2}\right)$$

This explains why the variation in frequency due to damping is negligible in most high- and moderate-$Q$ systems.

2 Driven oscillations

Let us analyze the effect of a periodic driving force, $F(t) = F_0 \cos(\omega t)$, on a damped oscillator. When the oscillator reaches steady state, the amplitude $A$ is given by

$$A = \frac{F_0/m}{\sqrt{(\omega^2 - \omega_0^2)^2 + (b/m)^2 \omega^2}}$$

After some algebraic manipulations and expressing $b/m$ in terms of the $Q$ factor,

$$A = \frac{F_0/m\omega_0^2}{\sqrt{(\omega^2/\omega_0^2 - 1)^2 + \frac{1}{Q^2} \cdot \omega^4/\omega_0^4}}$$
To make this expression easier to interpret, let us look at the expression in the numerator.

\[ \Delta L \equiv \frac{F_0}{m\omega_0^2} = \frac{F_0}{k} \]

This is how much the spring is stretched or compressed when a steady force of magnitude \( F_0 \) is applied to it. \( \Delta L \) could also be thought of as the amplitude \( A \) at a low (approaching zero) frequency of the driving force. Next, let us define a new variable \( x \) parameterizing the frequency so that

\[ x \equiv \frac{\omega}{\omega_0} \]

Now,

\[ A(x) = \frac{\Delta L}{\sqrt{(x^2 - 1)^2 + \frac{x^4}{Q^2}}} \]

This function has a maximum at a certain value of \( x \). Setting the derivative of \( A(x) \) to zero and solving for \( x_{\text{max}} \), for \( Q \gg 1 \),

\[ x_{\text{max}} = \frac{1}{\sqrt{1 + 1/Q^2}} \approx 1 - 1/2Q^2 \approx 1 \]

or, alternatively,

\[ \omega_{\text{max}} = \omega_0 \]

also known as the resonance condition. What is the amplitude at that frequency?

\[ A(x = 1) = Q \cdot \Delta L \]

meaning that the amplitude at the resonance frequency is \( Q \) times greater than the amplitude off resonance (at zero frequency).

How broad is the resonance, frequency-wise? In other words, what is the bandwidth of the resonance peak? Let us look at the width of the peak at the amplitude equal to \( A_{\text{max}}/\sqrt{2} \). The energy of the oscillation is proportional to the square of its amplitude, so we are looking for the full width at the half max level (FWHM) of energy. A few more algebraic steps yield the FWHM, \( \Delta x \) to be equal to \( 1/Q \) which translates into the bandwidth

\[ \Delta \omega = \omega_0/Q \]

Thus, \( Q \) factor could also be defined as the ratio of the central frequency \( \omega_0 \) to the bandwidth \( \Delta \omega \).

3 Summary

The main point of this write-up is to show how versatile the \( Q \) factor is in describing damped oscillations and resonance.

- \( Q \) is a measure of damping. It shows how many periods fit in one decay-time interval, or how many periods it takes for the oscillation to ring down, run out of energy.
- \( Q \) is a measure of how sharp the resonance is. The resonant amplitude is proportional to \( Q \), whereas the width is inversely proportional to \( Q \).
- \( Q \) is also a connecting link between the simple oscillator model and the model of a resonator, a system with a spectrum of natural frequencies which could be selectively excited by applying a periodic external disturbance.