Implicit vs. Explicit Functions:

The equation $y = x\sin x$ explicitly defines y as a function of x. Plug in a value of x on the right hand side and out pops y on the left hand side. We write $y = f(x)$ to denote explicit functions.

Consider the equation

$$x^2 + y^3 = 1$$

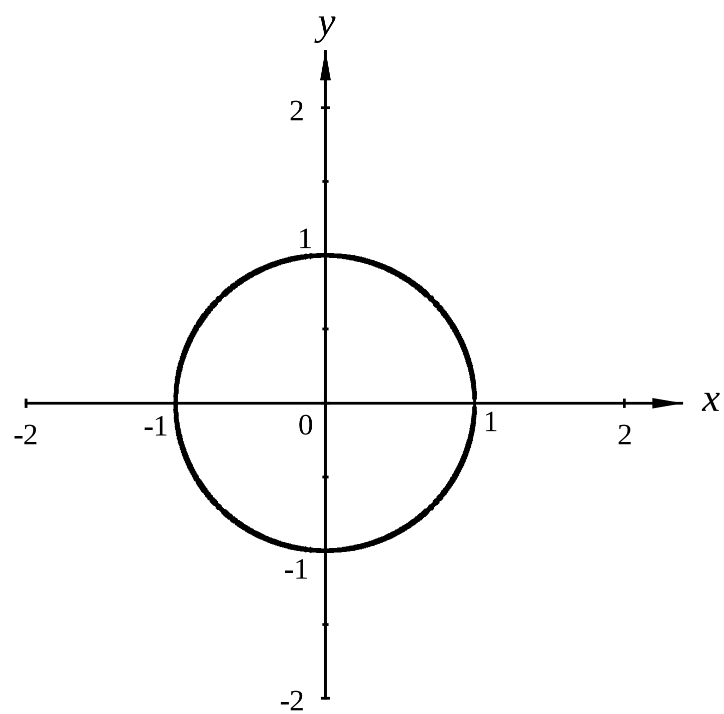
If $x = 0$, $y^3 = 1$, or $y = 1$.

If $x = 1$, $y^3 = 0$, or $y = 0$.

If $x = \sqrt{2}$, $2 + y^3 = 1$, or $y^3 = -1$, and so $y = -1$.

Any equation in *x* and *y* implicitly defines *y* as a function(s) of *x*.

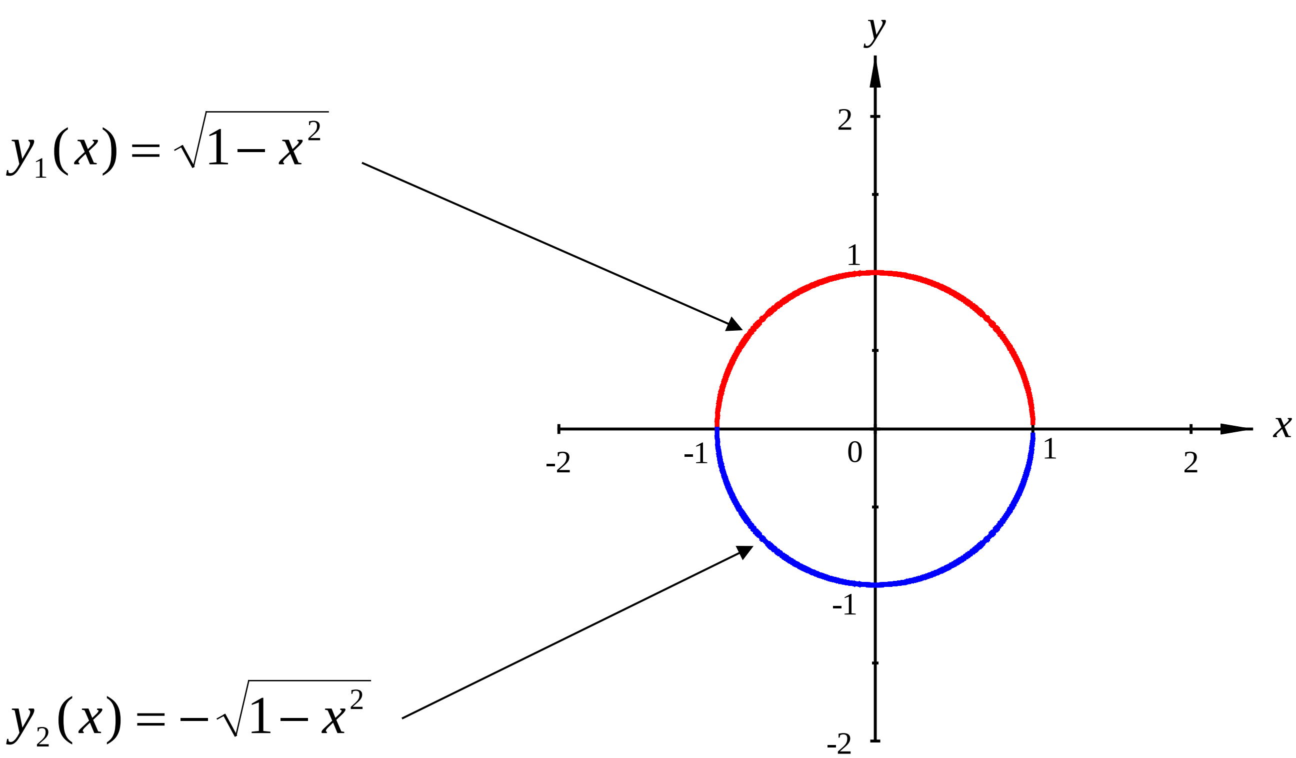
Example: Consider $x^2 + y^2 = 1$, which is a circle of radius 1, centered about the origin. Knowing the graph of the circle one immediately knows it fails to define a function in the usually sense since it does not pass the vertical line test.



In solving $x^2 + y^2 = 1$ for y it becomes apparent why this is the case.

$$x^2 + y^2 = 1 \quad \Rightarrow \quad y^2 = 1 - x^2 \quad \Rightarrow \quad y = \pm\sqrt{1 - x^2}.$$

Except for $x = \pm 1$, every value of x in the domain yields 2 values for y. Picking the "+" sign yields a value on the upper part of the circle and choosing the "–" sign gives the lower part of the circle.



While one doesn't view the graph of $x^2 + y^2 = 1$ as coming from a function, one can still wonder about lines tangent to the graph of the circle at points (a,b). The slope of these lines is viewed as the derivative

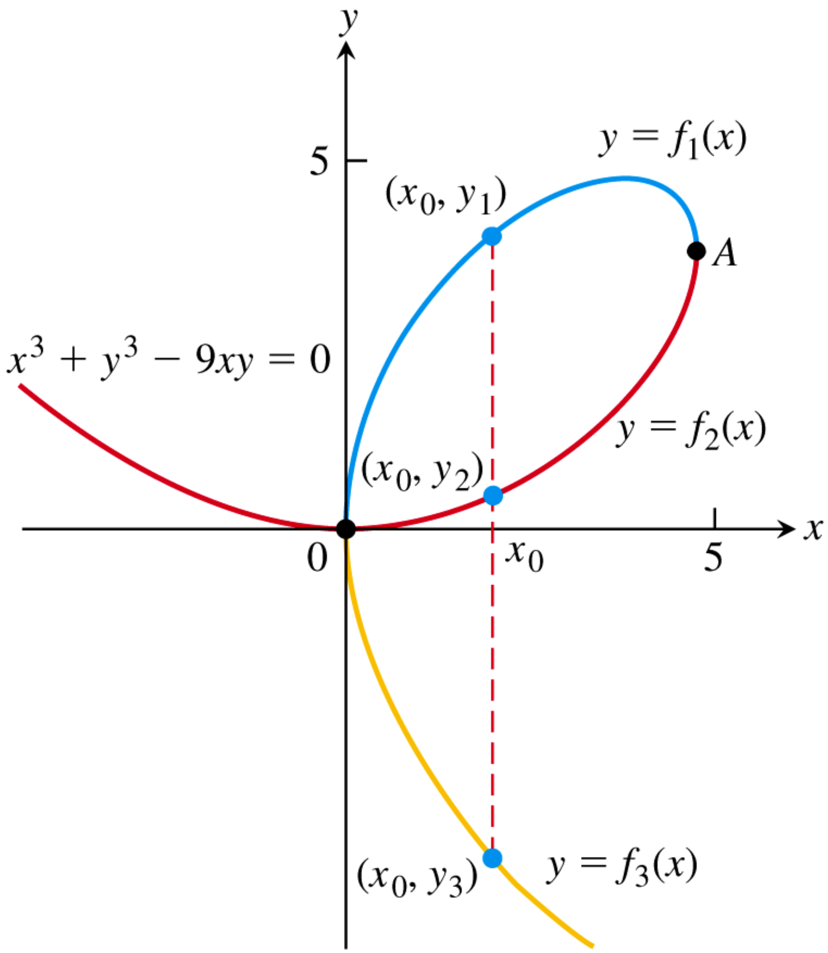
$$\left.\frac{dy}{dx}\right|_{(x,y)=(a,b)}$$

A general equation in *x* and *y* can be written as

$$F(x,y) = 0.$$

The set of solution points (*x*, *y*), yield the graph of this equation.

Since it isn't always possible to solve a given equation for *y* as an explicit function or functions of *x*, a method is needed to compute the derivative or slope of the graph at particular points.



y
5
$y = f_1(x)$
(x_0, y_1)
A
$x^3 + y^3 - 9xy = 0$
$y = f_2(x)$
(x_0, y_2)
x
0
x_0
5
(x_0, y_3)
$y = f_3(x)$

Implicit Differentiation

When trying to compute the derivative $\frac{dy}{dx}$, given an equation $F(x,y)=0$, the method is to view *y* as a function of *x*, and use the chain rule.

Example: Find the slope of the line tangent to the graph of the circle $x^2+y^2=1$ at points (x,y) on the circle.

Take the *x*-derivative of the equation assuming that $y=y(x)$.

$$\frac{d}{dx}(x^2+[y(x)]^2)=\frac{d}{dx}(1)$$

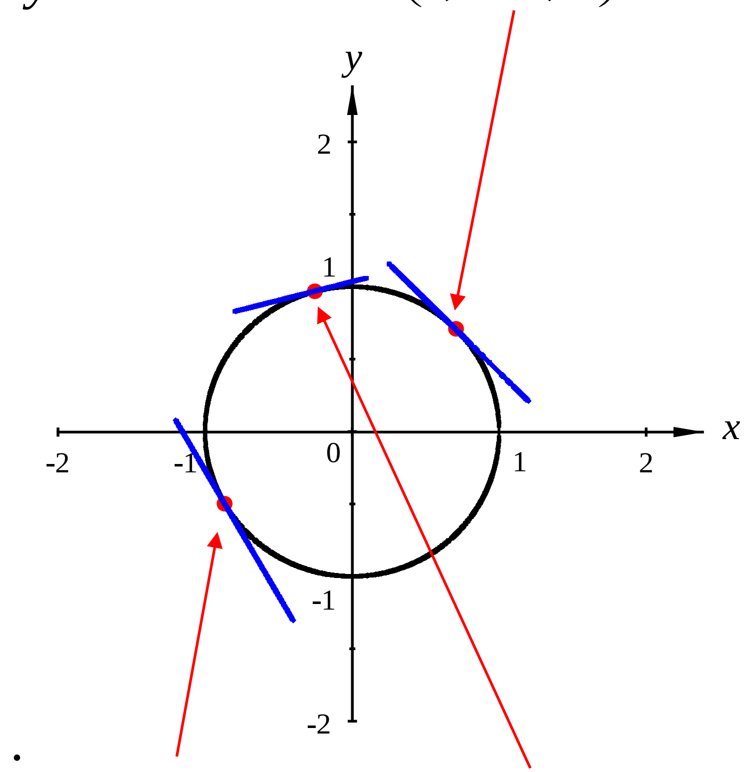
or

$$\frac{d}{dx}(x^2)+\frac{d}{dx}[y(x)]^2=0,$$

or

$$2x+2y\frac{dy}{dx}=0 \qquad \Rightarrow \qquad \frac{dy}{dx}=-\frac{x}{y}$$

With $\frac{dy}{dx} = -\frac{x}{y}$. At the point $\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$, $\frac{dy}{dx} = -\frac{1/\sqrt{2}}{1/\sqrt{2}} = -1$



Also shown are the points $\left(-\frac{\sqrt{3}}{2}, -\frac{1}{2}\right)$ and $\left(-\frac{1}{4}, \frac{\sqrt{15}}{4}\right)$.

Example: Using implicit differentiation determine the derivative $\frac{dy}{dx}$ at every point on the graph of

$$y^4 = x^3.$$

Take the x derivative of the equation assuming $y = y(x)$.

$$\frac{d}{dx}(y^4) = \frac{d}{dx}x^3$$

Using the chain rule on the LHS:

$$4y^3\frac{dy}{dx} = 3x^2$$

Solving for $\frac{dy}{dx}$ gives

$$\frac{dy}{dx} = \frac{3x^2}{4y^3}$$

Example: Determine the derivative $\frac{dy}{dx}$ at every point on the graph of

$$y^2 + x^3 + 2 = 0.$$

Take the x derivative of the equation assuming $y = y(x)$.

$$\frac{d}{dx}(y^2 + x^3 + 2) = \frac{d}{dx}0 = 0$$

$$\frac{d}{dx}(y^2 + x^3 + 2) = \frac{d}{dx}(y^2) + \frac{d}{dx}(x^3) = 2y\frac{dy}{dx} + 3x^2 = 0$$

$$2y\frac{dy}{dx} = -3x^2 \quad \text{or} \quad \frac{dy}{dx} = \frac{-3x^2}{2y}$$

Example: Determine the derivative $\dfrac{dy}{dx}$ at every point on the graph of

$$y^2x^2 = x+1.$$

Take the *x* derivative of the equation assuming $y = y(x)$.

$$\frac{d}{dx}(y^2x^2) = \frac{d}{dx}(x+1) = 1$$

Using the chain rule and product rule on the left hand side,

$$\frac{d}{dx}(y^2x^2) = \left(\frac{d}{dx}(y^2)\right)x + y^2\left(\frac{d}{dx}(x^2)\right) = 1$$

Or $$\left(2y\frac{dy}{dx}\right)x + y^2(2x) = 1 \text{ or } 2xy\frac{dy}{dx} + 2xy^2 = 1$$

$$2xy\frac{dy}{dx} = 1 - 2xy^2 \quad \text{or} \quad \frac{dy}{dx} = \frac{1-2xy^2}{2xy}$$

Example: Determine the derivative $\dfrac{dy}{dx}$ at every point on the graph of

$$y^3 + y^2 + y + x = 0.$$

Take the *x* derivative of the equation assuming $y = y(x)$.

$$\frac{d}{dx}(y^3 + y^2 + y + x) = 0$$

Using the chain rule,

$$\frac{d}{dx}(y^3 + y^2 + y + x) = \frac{d}{dx}(y^3) + \frac{d}{dx}(y^2) + \frac{d}{dx}(y) + \frac{d}{dx}(x) = 0$$

$$3y^2\frac{dy}{dx} + 2y\frac{dy}{dx} + \frac{dy}{dx} + 1 = 0$$

$$\frac{dy}{dx}(3y^2 + 2y + 1) = -1 \quad \text{or} \quad \frac{dy}{dx} = \frac{-1}{3y^2 + 2y + 1}$$

Example: Determine the derivative $\frac{dy}{dx}$ at every point on the graph of

$$\sin(y) = \sqrt{x}$$

$$\frac{d}{dx}\sin(y) = \frac{d}{dx}(\sqrt{x})$$

$$\cos(y)\frac{dy}{dx} = \frac{1}{2\sqrt{x}}$$

$$\frac{dy}{dx} = \frac{1}{2\cos y\sqrt{x}}$$

Proof of the power rule $\frac{d}{dx}x^r = r\,x^{r-1}$ for rational values of *r*.

Suppose $y = x^r$. If *r* is rational one can write $r = \frac{m}{n}$, where *m* and *n* are integers.

$$y = x^{m/n} \quad \Rightarrow \quad y^n = x^m$$

Taking the *x* derivative:

$$\frac{d}{dx}(y^n) = \frac{d}{dx}(x^m)$$

$$n\,y^{n-1}\frac{dy}{dx} = m\,x^{m-1}$$

$$\frac{dy}{dx} = \frac{m\,x^{m-1}}{n\,y^{n-1}} = \frac{m}{n}\frac{x^{m-1}}{(x^{m/n})^{n-1}} = \frac{m}{n}\frac{x^{m-1}}{x^{\frac{m(n-1)}{n}}} = \frac{m}{n}\frac{x^{m-1}}{x^{m-\frac{m}{n}}} = \frac{m}{n}x^{\frac{m}{n}-1} = rx^{r-1}$$