Exponential growth and decay applications

We wish to solve an equation that has a derivative.

\[
\frac{dy}{dx} = ky \quad k > 0
\]

This equation says that the rate of change of the function is proportional to the function.

The solution is \( y = ce^{kx} \). We can show this by taking the derivative of \( y \).

\[
\frac{dy}{dx} = \frac{d}{dx} (ce^{kx}) = c(ke^{kx}) = k(ce^{kx}) = ky
\]

Where \( k \) is either given or determined from the data and \( c \) is an arbitrary constant.
Suppose \( y \) is replaced by \( P \) which represents the population of some species. The assumption \( P' = kP \) makes perfect sense for smaller populations. It says the rate of change of the population is proportional to the population.

A specific problem is:

\[
P' = kP
\]

With initial condition

\[
P(0) = P_0
\]

which is called an initial value problem.

The initial condition states what the starting population is at time = 0.

**Example:** Suppose

\[
P' = 0.01P \text{ so } k = 0.01.
\]

Find the solution given

\[
P(0) = 100.
\]

We know the solution is \( P = ce^{0.01t} \). To get the value of \( c \), plug in 0. So,
\[ P(0) = ce^{0.01(0)} = ce^0 = c(1) = c = 100 \]

So the specific solution is

\[ P(t) = 100e^{0.01t} \]
If \( k \) is not specified some other piece of information is needed.

Suppose

\[
P' = kP
\]

\[
P(0) = 100
\]

The last equation again gives an initial population of 100 people.

If, in addition, we know

\[
P(1) = 1000
\]

We can figure out \( k \).

The solution from before is

\[
P(t) = 100e^{kt}
\]
To get $k$ plug in the second point $P(1) = 1000$

\[ P(1) = 100e^{k(1)} = 100e^k = 1000 \]  

or

\[ e^k = 10 \]

Solving for $k$ by taking the natural logarithm of both side.

\[ \ln(e^k) = \ln10 \implies \]

\[ k = \ln10 \]

And the solution is

\[ P(t) = 100e^{(\ln10)t} \]
Let \( P(t) = P_0 e^{kt} \), and suppose the extra piece of information is the time \( T \) when the initial population doubles. That is:

\[
P(T) = 2P_0
\]

i.e. The time \( T \) is also called the generation time. So we need to solve for \( T \):

\[
P_0 e^{kT} = 2P_0 \quad \text{or} \quad e^{kT} = 2
\]

Notice that the initial population is no long in the equation. Logging both sides gives

\[
\ln(e^{kT}) = \ln 2
\]

This is a crucial equation that relates the growth constant \( k \) to the doubling time \( T \).
The growth rate $k$ and the generation (doubling time) are linked by the formula

$$kT = \ln 2$$

Dividing by $T$ gives

$$k = \frac{\ln 2}{T}$$

Dividing by $k$ gives

$$T = \frac{\ln 2}{k}$$

**Example:** What is the growth rate $k$ if the doubling time is $T = 24.568$?

$$k = \frac{\ln 2}{24.568} \approx 0.0282$$

**Example:** What is the doubling time if the growth rate is 0.024.

$$T = \frac{\ln 2}{0.024} \approx 28.88$$
**Example:** Suppose a population has a doubling time $T = 17.38$ years, and an initial population of 2500. What is the population after 10 years.

First compute $k$: 

$$k = \frac{\ln 2}{17.38} \approx 0.0399.$$

Plugging into the solution equation gives

$$P(t) = 2500 e^{(0.0399)t}$$

$$P(10) = 2500 e^{(0.0399)10} = 2500 e^{0.399} \approx 3726$$

**Example:** A certain town has an initial population of 10,231 and it doubles in 130 years. Find the solution and determine when the population is 17,000.

So the solution is $P(t) = 10,231 e^{kt}$. To get $k$ we need to solve $k = \frac{\ln 2}{T}$ so

$$k = \frac{\ln 2}{130} \approx 0.005332$$
And so \[ P(t) = 10,231 e^{(0.005332)t} \]

Let \( t = t_0 \) be the time when the population is 17,000, then

\[ P(t_0) = 10,231 e^{(0.005332)t_0} = 17,000 \]

Which gives

\[ e^{(0.005332)t_0} = \frac{17,000}{10,231} \]

Logging both sides:

\[ (0.005332)t_0 = \ln \left( \frac{17,000}{10,231} \right) \]

Which gives

\[ t_0 = \frac{\ln \left( \frac{17,000}{10,231} \right)}{0.005332} \approx 95.23 \text{ years} \]
Logistic Population Model

The problem with the exponential solution we just obtained is that the population goes to $\infty$ as time goes to $\infty$. We all know that limited space and or food limits the population. A better model is called the Logistic model. This population model is

$$P(t) = \frac{L}{1 + be^{-kt}}$$

Notice that when $t = 0$, $P(0) = \frac{L}{1 + b} = \frac{L}{1 + b} = P_0$, and when $t$ goes to $\infty$,

$$\lim_{t \to \infty} P(t) = \lim_{t \to \infty} \frac{L}{1 + be^{-kt}} = \frac{L}{1 + \lim_{t \to \infty} be^{-kt}} = \frac{L}{1 + 0} = L$$

This means the limiting population is $L$. Depending on initial condition the solution looks like:
Notice that in the upper curve, the initial population is greater than $L$, and in the lower curve, the initial population is less than the limiting population.
Exponential Decay:

We again wish to solve an equation that has a derivative.

\[
\frac{dy}{dx} = -ky \quad k > 0
\]

This equation says that the rate of change of the function is proportional to the function, but now there is a negative on the right hand side.

The solution is \( y = ce^{-kx} \). We can show this by taking the derivative

\[
\frac{dy}{dx} = -cke^{-kx} = -k(ce^{-kx}) = -ky
\]

Where \( k \) is either given or determined from the data and \( c \) is an arbitrary constant determined by the initial condition.
Suppose $y$ is replaced by $N$ which represents the amount of radioactive material in some object. The assumption $N' = -kN$ makes sense. It says the rate of change of the amount is proportional to the amount present.

A specific problem is:

$$N' = -kN \quad k > 0$$

With initial condition

$$N(0) = N_0$$

Whose solution is

$$N(t) = N_0e^{-kt}$$
Example: Suppose

\[ N' = -0.052N \text{ then } k = 0.052. \]

Find the solution given

\[ N(0) = N_0. \]

We know the solution is \( N = ce^{-0.52t} \). To see this:

\[
N' = \frac{d}{dx}(ce^{-0.052t}) = c(-0.052e^{-0.052t}) = -c(ke^{-0.052t}) = -kN
\]

To get the value of \( c \), plug in 0. So,

\[ N(0) = ce^{-0.052(0)} = c = N_0 \]

So the specific solution is

\[ N(t) = N_0 e^{-0.052t} \]
If $k$ is not specified some other piece of information is needed.

Suppose

$$N' = -kN$$

$$N(0) = N_0 = 100$$

Which is the initial population. If in addition, we know

$$N(2) = 75$$

We can then figure out $k$.

First, the solution is

$$N(t) = N_0 e^{-kt} = 100 e^{-kt}$$
To get $k$ plug in the second point

$$
N(2) = 100e^{-k(2)} = 100e^{-2k} = 75 \text{ or }
100e^{-2k} = 75 \Rightarrow 
$$
$$
 e^{-2k} = .75 
$$

Solving for $k$ by taking the natural logarithm of both side.

$$
\ln(e^{-2k}) = \ln.75 \Rightarrow -2k = \ln.75 \Rightarrow k = \frac{\ln.75}{-2} > 0 
$$

Note: \( \ln(.75) = \ln(3/4) = \ln[(4/3)^{-1}] = -\ln(4/3) \)

And the solution is

$$
N(t) = 100e^{-(\frac{\ln(4/3)}{2})t} 
$$
If $N(t) = N_0 e^{-kt}$ and the extra piece of information is

$$N(T) = (1/2)N_0$$

Then we want the time $T$ when the initial amount halves.

$$N_0 e^{-kT} = (1/2)N_0$$

or

$$e^{-kT} = 1/2$$

Notice that the equation doesn’t have $N_0$ in it. Logging both sides gives

$$\ln(e^{-kT}) = \ln(1/2) = -\ln 2$$
$$-kT = -\ln 2$$
$$kT = \ln 2$$

This is just like the equation before. The decay constant $k$ is related to the halving time $T$, also called the half life.
The decay rate $k$ and the half life $T$, are related by

$$kT = \ln 2 \sim 0.693147$$

In which case

$$k = \frac{\ln 2}{T} \quad \text{and} \quad T = \frac{\ln 2}{k}$$

Example: What is the decay rate $k$ if the halving time if $T = 36.45$?

$$k = \frac{\ln 2}{36.45} \approx 0.019016$$

Example: What is the half-life if the growth rate is 0.024?

$$T = \frac{\ln 2}{0.024} \approx 28.88$$
Example: Carbon-14 has a half-life of 5750 years. Suppose an object has lost 20% of its carbon-14. How old is it?

The decay rate is

\[ k = \frac{\ln 2}{T} = \frac{\ln 2}{5750} \approx 0.000125 \]

So the amount present is

\[ N(t) = N_0 e^{-0.000125t} \]

So we want to know how old (the time \( t_0 \)) when there is 80% left.

\[ N(t_0) = N_0 e^{-0.000125t_0} = 0.8N_0 \]

Solving for \( t_0 \)

\[ e^{-0.000125t_0} = 0.8 \]

so

\[ t_0 = \frac{\ln 0.8}{-0.000125} \approx 1785 \text{ yrs} \]
Newton’s Law of Cooling.

Suppose the temperature of an object changes at a rate proportional to the difference of the object’s temperature and the surrounding medium.

\[ T'(t) = -k(T - C) \]

with \( T(0) = T_0 \)

Notice. If \( T > C \) then \( T'(t) < 0 \) and the object cools. If \( T < C \) then \( T'(t) > 0 \) then the object warms up.
Suppose we let 

\[ P(t) = T(t) - C \]

Then 

\[ T'(t) = P'(t) = -k(T(t) - C) = -kP(t) \]

Or 

\[ P'(t) = -kP(t) \]

Which we just solved so 

\[ P(t) = P_0 e^{-kt} \]

Then 

\[ P(0) = P_0 = T_0 - C \]

And then 

\[ T(t) = (T_0 - C)e^{-kt} + C \]

Suppose the object is initially 750 degrees, and the surrounding medium is 250 degrees. Then
\[ T(t) = 500e^{-kt} + 250 \]

Which has graph:

\[ T(t) \]

\[ T_0 = 750^\circ \]

\[ C = 250^\circ \]