Exponential growth and decay applications

We wish to solve an equation that has a derivative.

$$\frac{dy}{dx} = ky \quad k > 0$$

This equation says that the rate of change of the function is proportional to the function.

The solution is $y = ce^{kx}$. We can show this by taking the derivative of *y*.

$$\frac{dy}{dx} = \frac{d}{dx}(ce^{kt}) = c(ke^{kx}) = k(ce^{kx}) = ky$$

Where *k* is either given or determined from the data and *c* is an arbitrary constant.

Suppose y is replaced by P which represents the population of some species. The assumption $P' = kP$ makes perfect sense for smaller populations. It says the rate of change of the population is proportional to the population.

A specific problem is:

$$P' = kP$$

With initial condition

$$P(0) = P_0$$

which is called an initial value problem.

The initial condition states what the starting population is at time = 0.

Example: Suppose

$$P' = 0.01P \text{ so } k = 0.01.$$

Find the solution given

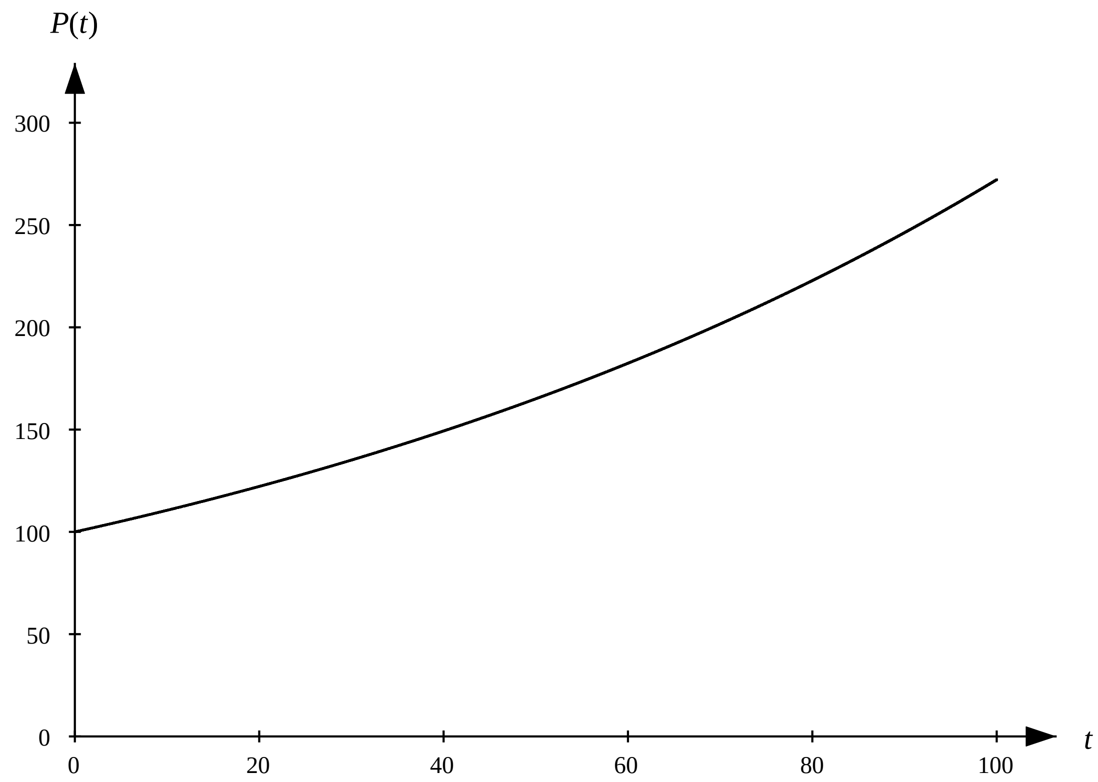
$$P(0) = 100.$$

We know the solution is $P = ce^{0.01t}$. To get the value of c, plug in 0. So,

$$P(0) = ce^{0.01(0)} = ce^{0} = c(1) = c = 100$$

So the specific solution is

$$P(t) = 100e^{0.01t}$$



If k is not specified some other piece of information is needed.

Suppose

$$P' = kP$$

$$P(0) = 100$$

The last equation again gives an initial population of 100 people.

If, in addition, we know

$$P(1) = 1000$$

We can figure out k.

The solution from before is

$$P(t) = 100\,e^{kt}$$

To get k plug in the second point $P(1) = 1000$

$$P(1) = 100e^{k(1)} = 100e^{k} = 1000 \quad \text{or}$$

$$\boxed{e^{k} = 10}$$

Solving for k by taking the natural logarithm of both side.

$$\ln(e^{k}) = \ln 10 \Rightarrow$$

$$k = \ln 10$$

And the solution is

$$P(t) = 100\, e^{(\ln 10)t}$$

Let $P(t) = P_0 e^{kt}$, and suppose the extra piece of information is the time T when the initial population doubles. That is:

$$P(T) = 2P_0$$

i.e. The time T is also called the generation time. So we need to solve for T:

$$P_0 e^{kT} = 2P_0 \quad \text{or} \quad e^{kT} = 2$$

Notice that the initial population is no long in the equation.

Logging both sides gives $\ln(e^{kT}) = \ln 2$

$$\boxed{kT = \ln 2}$$

This is a crucial equation that relates the growth constant k to the doubling time T.

The growth rate k and the generation (doubling time) are linked by the formula

$$kT = \ln 2$$

Dividing by T gives

$$k = \frac{\ln 2}{T}$$

Dividing by k gives

$$T = \frac{\ln 2}{k}$$

Example: What is the growth rate k if the doubling time is $T = 24.568$?

$$k = \frac{\ln 2}{24.568} \approx 0.0282$$

Example: What is the doubling time if the growth rate is 0.024.

$$T = \frac{\ln 2}{0.024} \approx 28.88$$

Example: Suppose a population has a doubling time $T = 17.38$ years, and an initial population of 2500. What is the population after 10 years.

First compute k: $k = \dfrac{\ln 2}{17.38} \approx 0.0399.$

Plugging into the solution equation gives

$$P(t) = 2500\,e^{(0.0399)t}$$

$$P(10) = 2500e^{(0.0399)\cdot 10} = 2500\,e^{0.399} \approx 3726$$

Example: A certain town has an initial population of 10,231 and it doubles in 130 years. Find the solution and determine when the population is 17,000.

So the solution is $P(t) = 10{,}231e^{kt}$. To get k we need to solve $k = \dfrac{\ln 2}{T}$ so

$$k = \frac{\ln 2}{130} \approx 0.005332$$

And so $$P(t) = 10{,}231 e^{(0.005332)t}$$

Let $t = t_0$ be the time when the population is 17,000, then

$$P(t_0) = 10{,}231 e^{(0.005332)t_0} = 17{,}000$$

Which gives

$$e^{(0.005332)t_0} = \frac{17{,}000}{10{,}231}$$

Logging both sides:

$$(0.005332)t_0 = \ln\left(\frac{17{,}000}{10{,}231}\right)$$

Which gives

$$t_0 = \frac{\ln\left(\frac{17{,}000}{10{,}231}\right)}{0.005332} \approx 95.23 \text{years}$$

Logistic Population Model

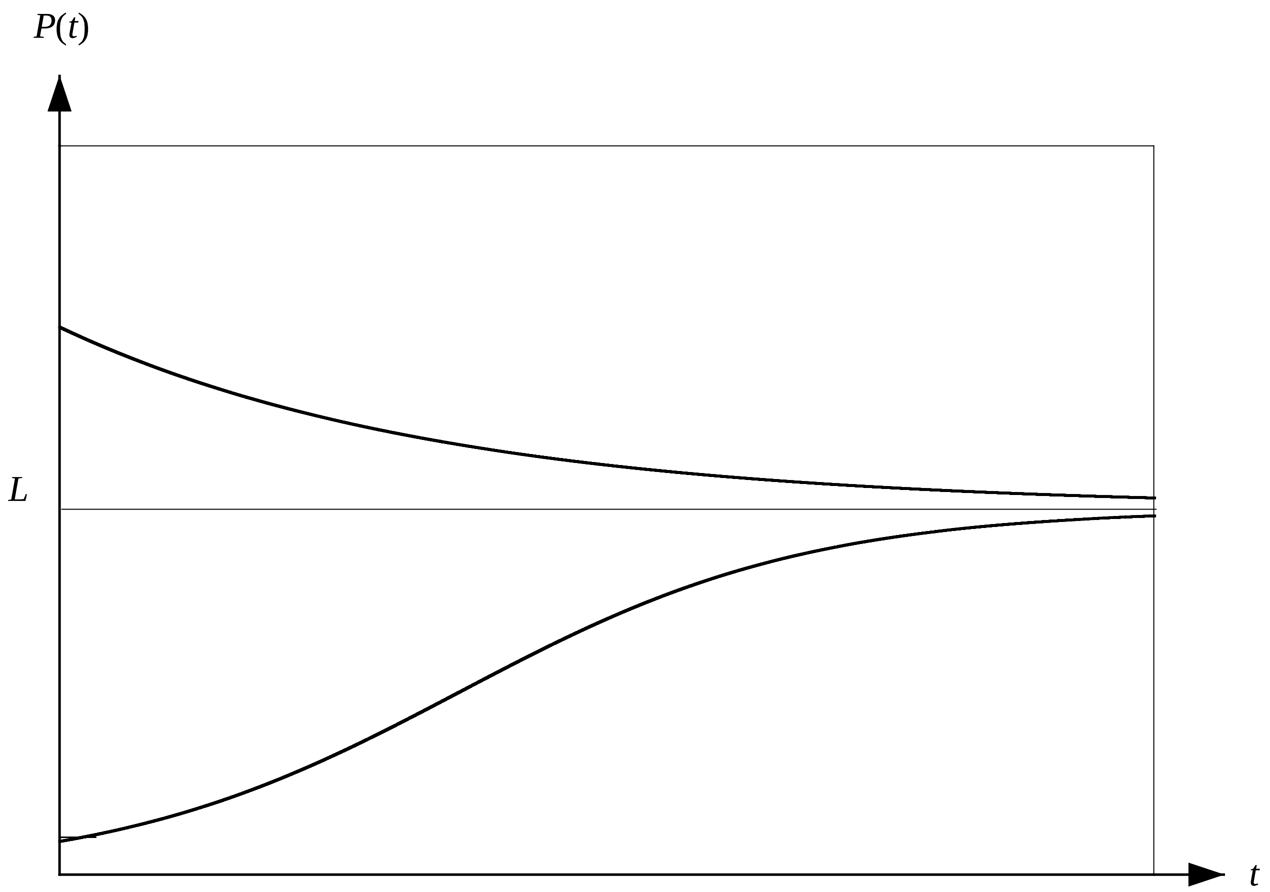
The problem with the exponential solution we just obtained is that the population goes to ∞ as time goes to ∞. We all know that limited space and or food limits the population. A better model is called the Logistic model. This population model is

$$P(t) = \frac{L}{1 + be^{-kt}}$$

Notice that when $t = 0$, $P(0) = \frac{L}{1+be^0} = \frac{L}{1+b} = P_0$, and when t goes to ∞,

$$\lim_{t\to\infty} P(t) = \lim_{t\to\infty} \frac{L}{1+be^{-kt}} = \frac{L}{1+\lim\limits_{t\to\infty} be^{-kt}} = \frac{L}{1+0} = L$$

This means the limiting population is L. Depending on initial condition the solution looks like:



Notice that in the upper curve, the initial population is greater than L, and in the lower curve, the initial population is less than the limiting population

Exponential Decay:

We again wish to solve an equation that has a derivative.

$$\frac{dy}{dx} = -ky \quad k > 0$$

This equation says that the rate of change of the function is proportional to the function, but now there is a negative on the right hand side.

The solution is $y = ce^{-kx}$. We can show this by taking the derivative

$$\frac{dy}{dx} = -cke^{-kx} = -k(ce^{-kx}) = -ky$$

Where k is either given or determined from the data and c is an arbitrary constant determined by the initial condition.

Suppose y is replaced by N which represents the amount of radioactive material in some object. The assumption $N' = -kN$ makes sense. It says the rate of change of the amount is proportional to the amount present.

A specific problem is:

$$N' = -kN \quad k > 0$$

With initial condition

$$N(0) = N_0$$

Whose solution is

$$N(t) = N_0 e^{-kt}$$

Example: Suppose

$$N' = -0.052N \text{ then } k = 0.052.$$

Find the solution given

$$N(0) = N_0.$$

We know the solution is $N = ce^{-0.52t}$. To see this:

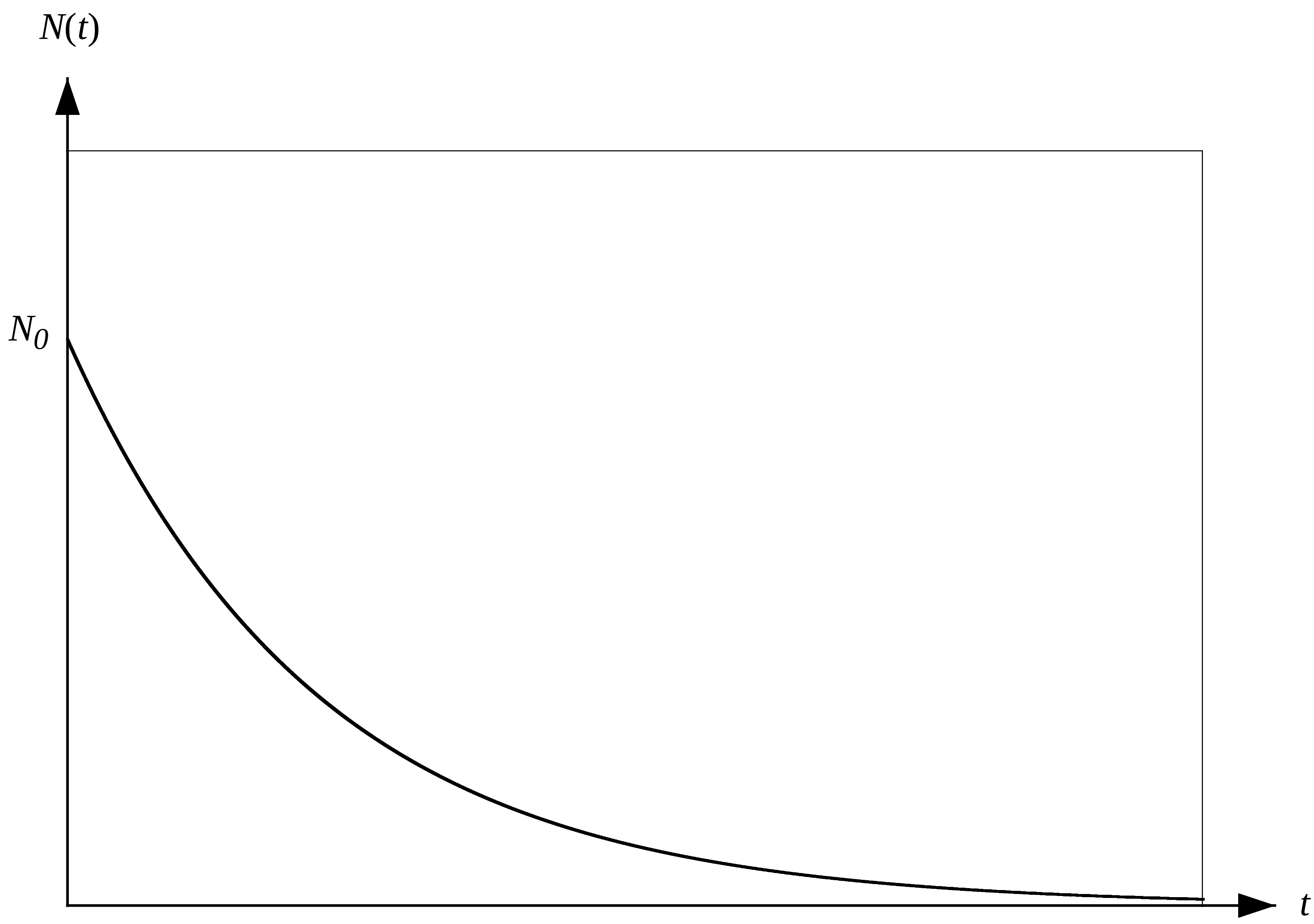
$$N' = \frac{d}{dx}(ce^{-0.052t}) = c(-0.052e^{-0.052t}) = -c(ke^{-0.052t}) = -kN$$

To get the value of *c*, plug in 0. So,

$$N(0) = ce^{-0.052(0)} = c = N_0$$

So the specific solution is

$$N(t) = N_0 e^{-0.052t}$$



N(t)
N0
t

If k is not specified some other piece of information is needed.

Suppose

$$N' = -kN$$

$$N(0) = N_0 = 100$$

Which is the initial population. If in addition, we know

$$N(2) = 75$$

We can then figure out k.

First, the solution is

$$N(t) = N_0 e^{-kt} = 100\, e^{-kt}$$

To get k plug in the second point

$$N(2) = 100e^{-k(2)} = 100e^{-2k} = 75 \text{ or}$$

$$100e^{-2k} = 75 \Rightarrow$$

$$e^{-2k} = .75$$

Solving for k by taking the natural logarithm of both side.

$$\ln(e^{-2k}) = \ln.75 \qquad \Rightarrow \qquad -2k = \ln.75 \qquad \Rightarrow \qquad k = \frac{\ln.75}{-2} > 0$$

Note: $\ln(.75) = \ln(3/4) = \ln[(4/3)^{-1}] = -\ln(4/3)$

And the solution is

$$N(t) = 100\,e^{-\left(\frac{\ln 4/3}{2}\right)t}$$

If $N(t) = N_0\, e^{-kt}$ and the extra piece of information is

$$N(T) = (1/2)N_0$$

Then we want the time T when the initial amount halves.

$$N_0\, e^{-kT} = (1/2)N_0$$

or

$$e^{-kT} = 1/2$$

Notice that the equation doesn't have N_0 in it. Logging both sides gives

$$\ln(e^{-kT}) = \ln(1/2) = -\ln 2$$

$$-kT = -\ln 2$$

$$kT = \ln 2$$

This is just like the equation before. the decay constant k is related to the halving time T, also called the half life.

The decay rate k and the half life T, are related by

In which case

$$kT = \ln 2 \sim 0.693147$$

$$k = \frac{\ln 2}{T} \text{ and } T = \frac{\ln 2}{k}$$

Example: What is the decay rate k if the halving time if $T = 36.45$?

$$k = \frac{\ln 2}{36.45} \approx 0.019016$$

Example: What is the half-life if the growth rate is 0.024?

$$T = \frac{\ln 2}{0.024} \approx 28.88$$

Example: Carbon-14 has a half-life of 5750 years. Suppose an object has lost 20% of its carbon-14. How old is it?

The decay rate is

$$k = \frac{\ln 2}{T} = \frac{\ln 2}{5750} \approx 0.000125$$

So the amount present is

$$N(t) = N_0 \, e^{-0.000125t}$$

So we want to know how old (the time t_0) when there is 80% left.

$$N(t_0) = N_0 \, e^{-0.000125t_0} = 0.8N_0$$

Solving for t_0

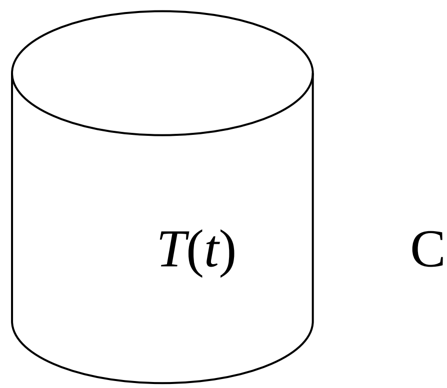
$$e^{-0.000125t_0} = 0.8$$

so

$$t_0 = \frac{\ln 0.8}{-0.000125} \approx 1785 yrs$$

Newton's Law of Cooling.

Suppose the temperature of an object changes at a rate proportional to the difference of the object's temperature and the surrounding medium.



The equation to solve is

$$T'(t) = -k(T - C)$$

$$\text{with } T(0) = T_0$$

Notice. If $T > C$ then $T'(t) < 0$ and the object cools. If $T < C$ then $T'(t) > 0$ then the object warms up.

Suppose we let

$$P(t) = T(t) - C$$

Then

$$T'(t) = P'(t) = -k(T(t) - C) = -kP(t)$$

Or

$$P'(t) = -kP(t)$$

Which we just solved so

$$P(t) = P_0 e^{-kt}$$

Then

$$P(0) = P_0 = T_0 - C$$

And then

$$T(t) = (T_0 - C)e^{-kt} + C$$

Suppose the object is initially 750 degrees, and the surrounding medium is 250 degrees. Then

$$T(t) = 500e^{-kt} + 250$$

Which has graph: